Math 345, Fall 2024

Solutions to Homework #13

Problem 1. IV.5, #2. Let f be an entire function. Suppose that there is a disk $D = D(a, r)$ — that is, the open disk centered at some point $a \in \mathbb{C}$ of some (positive) radius $r > 0$ — such that f does not attain any values in D. (That is, for all $z \in \mathbb{C}$, we have $f(z) \notin D$.) Prove that f is constant.

Proof. Let $h(z) = 1/(z-a)$, which is an analytic function on $\mathbb{C} \setminus \{a\}$. Since $f(z) \neq a$ for all $z \in \mathbb{C}$, we have $h(f(z))$ is analytic at all $z \in \mathbb{C}$; that is, $h(f(z)) = \dfrac{1}{f(z)-a}$ is entire.

Let $M = 1/r \in (0, \infty)$. Then for all $z \in \mathbb{C}$, we have $f(z) \notin D(a, r)$, so that $|f(z) - a| \geq r$. Therefore, $|h(f(z))| = \dfrac{1}{|f(z)-a|} \leq M$. That is, $h \circ f$ is bounded by M.

Since $h \circ f$ is a bounded entire function, it is constant by Liouville's Theorem. Call this constant $c \in \mathbb{C}$. Then for all $z \in \mathbb{C}$, we have $f(z) - a = 1/c$, and hence $f(z) = a + 1/c$. That is, $f(z)$ is the constant function $a + 1/c$. QED

Problem 2. IV.6, #2. Fix real numbers $a < b$, and let $h : [a, b] \to \mathbb{C}$ be continuous. The *Fourier transform* of h is the function $H : \mathbb{C} \to \mathbb{C}$ given by

$$H(z) = \int_a^b h(t)e^{-itz}\, dt.$$

Prove that H is an entire function, and that there are some positive constants $A, C > 0$ so that

$$\big|H(z)\big| \leq Ce^{A|y|} \qquad \text{for all } z = x + iy \in \mathbb{C}.$$

Proof. Define $f : [a, b] \times \mathbb{C} \to \mathbb{C}$ by $f(t, z) = h(t)e^{-itz}$, which is a product of continuous functions and hence is continuous. In addition, for any fixed $t \in [a, b]$, the function $z \mapsto f(t, z)$ is analytic

Therefore, by the Theorem at the top of page 121 (which was also on Video 15), the function $H(z)$ is analytic on $\mathbb{C}$. That is, H is entire.

Since h is continuous (and so is the absolute value function), the compositionn $t \mapsto |h(t)|$ is also continuous on $[a, b]$. Because $[a, b]$ is compact, it follows that $|h(t)|$ attains a maximum value B on $[a, b]$. [This is the Theorem on page 39, but for the closed interval $[a, b]$, it's also from Calc 1.] If $B = 0$, then increase B to 1, so that $B > 0$.

The segment $[a, b]$ is a path of length $L = b - a > 0$.
Define $\boxed{C = BL = B(b-a) > 0}$ and $\boxed{A = \max\{|a|, |b|\} > 0}$

Then for any $z = x + iy \in \mathbb{C}$, we have

$$\big|f(t, z)\big| = \big|h(t)\big| \cdot \Big|e^{-it(x+iy)}\Big| \leq B \cdot e^{ty} \leq Be^{A|y|},$$

since $ty \leq |t| \cdot |y|$ and since $|t| \leq A$. Thus, with $M = Be^{A|y|}$, the ML-estimate gives us

$$\big|H(z)\big| = \left|\int_a^b h(t)e^{-itz}\, dt\right| \leq Be^{A|y|} \cdot (b-a) = Ce^{A|y|}$$ QED

Problem 3. V.1, #5. It is a fact that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to some real number S. Prove that the series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

(which is just a rearrangement of the first series) converges to $3S/2$.

Proof. The first series may be written as $\sum_{k=1}^{\infty} a_k$, where $a_k = \frac{1}{4k-3} - \frac{1}{4k-2} + \frac{1}{4k-1} - \frac{1}{4k}$, and the second series may be written as $\sum_{k=1}^{\infty} b_k$, where

$$b_k = \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} = a_k + \frac{1}{4k-2} + \frac{1}{4k} - \frac{1}{2k} = a_k + \frac{1}{4k-2} - \frac{1}{4k}.$$

Thus, the $3n$-th partial sum of the second series is the n-th partial sum of $\sum b_k$, which is

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} \Big(\frac{1}{4k-2} - \frac{1}{4k}\Big) = \sum_{k=1}^{n} a_k + \frac{1}{2} \sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j}.$$

Taking the limit $n \to \infty$, we have $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = S + \frac{S}{2} = \frac{3S}{2}$. QED

[Well, technically that's only dealing with partial sums of $3n$ terms. But having shown that $3n$-th partial sums of the second converge to $3S/2$, we know *all* of its partial sums converge to $3S/2$, since the $(3n+1)$-st and $(3n+2)$-nd partial sums differ from the $3n$-th by $1/(4n+1)$ and by $[1/(4n+1) + 1/(4n+3)]$, respectively, both of which approach zero as $n \to \infty$.]

Problem 4. V.2, #10. Let $E_1, \ldots, E_n$ be subsets of $\mathbb{C}$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions that converges uniformly on each of the sets E_j, for $j = 1, \ldots, n$.
Prove that the sequence of functions also converges uniformly on E, where $E = E_1 \cup E_2 \cup \cdots \cup E_n$.

Proof. Given $\varepsilon > 0$, by the uniform convergence on each E_j, there are integers $N_1, N_2, \ldots, N_n \geq 1$ with the following property: for each $j = 1, \ldots, N$ every $k \geq N_j$, and every $z \in E_j$, we have $|f_k(z) - f(z)| < \varepsilon$.
Let $N = \max\{N_1, \ldots, N_n\}$. Given $k \geq N$ and $z \in E$, we have $z \in E_j$ for some $j = 1, \ldots, n$. Since $k \geq N \geq N_j$, then we have $|f_k(z) - f(z)| < \varepsilon$. QED