Math 345, Fall 2024

Solutions to Homework #12

Problem 1. IV.2, #1(a,b), variant. Let γ be *any* piecewise-smooth path in the plane from $-\pi i$ to πi. Use primitives to evaluate the following integrals.

$$\text{(a)} \int_\gamma z^4\,dz \qquad\qquad \text{(b)} \int_\gamma e^z\,dz$$

Solutions. **(a)**: Define $F(z) = \frac{1}{5}z^5$, so that $F'(z) = z^4$, and hence F is a primitive for z^4 on $\mathbb{C}$. Thus,

$$\int_\gamma z^4\,dz = F(\pi i) - F(-\pi i) = \frac{1}{5}(\pi i)^5 - \frac{1}{5}(-\pi i)^5 = \frac{\pi^5 i}{5}(1-(-1)) = \boxed{\frac{2\pi^5 i}{5}}$$

(a): Define $F(z) = e^z$, so that $F'(z) = e^z$, and hence F is a primitive for e^z on $\mathbb{C}$. Thus,

$$\int_\gamma e^z\,dz = F(\pi i) - F(-\pi i) = e^{\pi i} - e^{-\pi i} = -1-(-1) = \boxed{0}$$

Problem 2. IV.2, #2. Let γ_1 be *any* piecewise-smooth path in the right half-plane from $-\pi i$ to πi, and let γ_2 be *any* piecewise-smooth path in the left half-plane from $-\pi i$ to πi. For each path γ_j, choose an explicit primitive of $1/z$ (on the right half-plane, and on the left half-plane, respectively). Use this primitive to evaluate $\int_{\gamma_j} \frac{1}{z}\,dz$ for $j = 1, 2$.

Solutions. For γ_1, define $F_1(z) = \operatorname{Log} z$, which is analytic on the usual slit plane $\mathbb{C} \smallsetminus (-\infty, 0]$ with $F_1'(z) = 1/z$. That is, F_1 is a primitive for $1/z$ on this domain, which contains the right half-plane and hence contains γ_1. Note that

$$F_1(\pi i) = \log\pi + i\operatorname{Arg}(\pi i) = \log\pi + \frac{\pi i}{2} \quad\text{and}\quad F_1(-\pi i) = \log\pi + i\operatorname{Arg}(-\pi i) = \log\pi - \frac{\pi i}{2}.$$

Thus, $\displaystyle\int_{\gamma_1} \frac{1}{z}\,dz = F_1(\pi i) - F_1(-\pi i) = \log\pi + \frac{\pi i}{2} - \left(\log\pi - \frac{\pi i}{2}\right) = \boxed{\pi i}$

For γ_2, define $F_2(z)$ to be the branch of $\log z = \log|z| + i\arg(z)$ given by $0 < \arg(z) < 2\pi$. Then F_2 is analytic on the different slit plane $\mathbb{C} \smallsetminus [0, \infty)$ — that is, remove the *positive* real axis — again with $F_2'(z) = 1/z$. That is, F_2 is a primitive for $1/z$ on this domain, which contains the left half-plane and hence contains γ_2. Note that

$$F_2(\pi i) = \log\pi + i\arg(\pi i) = \log\pi + \frac{\pi i}{2} \quad\text{and}\quad F_2(-\pi i) = \log\pi + i\arg(-\pi i) = \log\pi + \frac{3\pi i}{2}.$$

Thus, $\displaystyle\int_{\gamma_2} \frac{1}{z}\,dz = F_2(\pi i) - F_2(-\pi i) = \log\pi + \frac{\pi i}{2} - \left(\log\pi + \frac{3\pi i}{2}\right) = \boxed{-\pi i}$

Problem 3. IV.3, #4. Use Cauchy's Theorem to prove the key step of the Fundamental Theorem of Algebra — any polynomial with no roots in $\mathbb{C}$ must be constant — using the following strategy. Let $P(z)$ be a polynomial with coefficients in $\mathbb{C}$ that is not constant. (That is, $P(z) \in \mathbb{C}[z]$ with $\deg(P) \geq 1$.) Write $P(z) = P(0) + zQ(z)$ for an appropriate polynomial $Q(z)$, and consider the integral

$$\oint_{|z|=R} \frac{1}{z}\,dz = \oint_{|z|=R} \frac{P(z)}{zP(z)}\,dz = \oint_{|z|=R} \frac{P(0)+zQ(z)}{zP(z)}\,dz = \oint_{|z|=R} \frac{P(0)}{zP(z)}\,dz + \oint_{|z|=R} \frac{Q(z)}{P(z)}\,dz.$$

On the one hand, we can compute the integral on the left side. On the other hand, if P has no roots, you can take the limit as $R \to \infty$ and use the ML-estimate and Cauchy's Theorem to bound the integrals on the right side. So if P has no roots, deduce a contradiction.

Proof. Assume P has no zeros in $\mathbb{C}$. Writing $P(z) = a_0 + a_1 z + \cdots + a_d z^d$ with $d \geq 1$ and $a_d \neq 0$, let $Q(z) = a_1 + a_2 z^2 + \cdots + a_d z^{d-1}$, so that $P(z) = zQ(z) + a_0$. [And of course $a_0 = P(0)$.]
For any real number $R > 0$, recall from direct computation (in the book, and also in class) that $\int_{|z|=R} \frac{1}{z}\, dz = 2\pi i$. Thus, $2\pi i = \int_{|z|=R} \frac{1}{z}\, dz = \int_{|z|=R} \frac{P(z)}{zP(z)}\, dz = \int_{|z|=R} \frac{a_0}{zP(z)}\, dz + \int_{|z|=R} \frac{Q(z)}{P(z)}\, dz.$
Consider the first integral on the right side. For $|z| = R$, we have

$$\frac{|zP(z)|}{R^{d+1}} = \frac{1}{R^{d+1}} |a_d z^{d+1} + a_{d-1} z^d + \cdots + a_0 z| \geq |a_d| - \sum_{j=0}^{d-1} |a_j| R^{j-d},$$

which approaches $|a_d| > 0$ as $R \to \infty$. Therefore, for R large enough, we have $|zP(z)| \geq |a_d/2| R^{d+1}$. For such R, then, and for $|z| = R$, the integrand of the first integral has $\left|\frac{a_0}{zP(z)}\right| \leq \frac{|2a_0|}{|a_d| R^{d+1}}$. Meanwhile, its path has length $2\pi R$. Hence, by the ML-estimate, for R large enough, we have $\left|\int_{|z|=R} \frac{a_0}{zP(z)}\, dz\right| \leq \frac{4\pi |a_0|}{|a_d| R^d}$.
The second integral is even simpler. Since P has no zeros, the integrand $Q(z)/P(z)$ is analytic on $\mathbb{C}$, and hence $\int_{|z|=R} Q(z)/P(z)\, dz = 0$ by Cauchy's Theorem.

Combining the above computations, we have $2\pi = |2\pi i| = \left|\int_{|z|=R} \frac{a_0}{zP(z)}\, dz\right| \leq \frac{4\pi |a_0|}{|a_d| R^d}$ for all R large enough. Taking the limit as $R \to \infty$, then, we have $2\pi \leq 0$, a contradiction. QED

Problem 4. IV.4, #1(a,b). Evaluate these integrals using the Cauchy Integral Formula and/or Cauchy Differentiation Formula and/or Cauchy's Theorem.

(a) $\oint_{|z|=2} \frac{z^n}{z-1}\, dz$ for each integer $n \geq 0$ (b) $\oint_{|z|=1} \frac{z^n}{z-2}\, dz$ for each integer $n \geq 0$

Solutions. **(a)**: The path bounds the disk $D(0,2)$. Let $f(z) = z^n$, which is analytic on this disk (and its boundary). Since $z = 1$ is inside the disk, the Cauchy Integral Formula for $f(1)$ gives us
$\oint_{|z|=2} \frac{z^n}{z-1}\, dz = 2\pi i f(1) = \boxed{2\pi i}$

(b): The path bounds the disk $D(0,1)$. Let $f(z) = z^n/(z-2)$, which is analytic on this disk (and its boundary), since $z = 2$ does **not** lie in this disk. So by Cauchy's Theorem, $\oint_{|z|=1} \frac{z^n}{z-2}\, dz = \boxed{0}$

Problem 5. IV.4, #1(e,g). Evaluate these integrals using the Cauchy Integral Formula and/or Cauchy Differentiation Formula and/or Cauchy's Theorem.

(e) $\oint_{|z|=1} \frac{e^z}{z^m}\, dz$ for each integer $m \in \mathbb{Z}$ (g) $\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$

Solutions. **(e)**: The path bounds the disk $D(0,1)$. If $m \leq 0$, let $f(z) = z^{-m} e^z$, which is analytic on this disk (and its boundary). As in part (b), then, by Cauchy's Theorem, the integal is 0.
On the other hand, if $m \geq 1$, let $f(z) = e^z$. Since $z = 0$ is inside the disk, the Cauchy Differentiation Formula gives us $\oint_{|z|=1} \frac{e^z}{z^m}\, dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(0) = \frac{2\pi i}{(m-1)!}$.

Thus, the full answer is $\boxed{\oint_{|z|=1} \frac{e^z}{z^m}\,dz = \begin{cases} \dfrac{2\pi i}{(m-1)!} & \text{if } m \geq 1, \\ 0 & \text{if } m \leq 0. \end{cases}}$

(g): The path bounds the disk $D(0,1)$. Let $f(z) = e^{-z}/(z^2-4)$, which is analytic on this disk (and its boundary), since the only roots $z = \pm 2$ of the denominator lie outside. We compute

$$f'(z) = \frac{-e^{-z}(z^2-4) - e^{-z}(2z)}{(z^2-4)^2}, \quad \text{and hence} \quad f'(0) = \frac{-(0-4)-0}{(0-4)^2} = \frac{1}{4}.$$

So by the Cauchy Differentiation Formula for $f'(0)$,

$$\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = \oint_{|z|=1} \frac{f(z)}{z^2}\,dz = \frac{2\pi i}{1!} f'(0) = \boxed{\frac{\pi i}{2}}$$