Math 345, Fall 2024

Solutions to Homework #11

Problem 1. III.2, #5. Fix $0 < a < b$, and let $D = \{z \in \mathbb{C} : a < |z| < b\} \smallsetminus (-b, -a)$, which is a slit annulus. Given points $A, B \in D$ and two paths γ_0, γ_1 from A to B in D, give an explicit homotopy from γ_0 to γ_1 in D.

Solution/Proof. Note that D can be rewritten as $D = \{z \in \mathbb{C} : a < |z| < b \text{ and } -\pi < \operatorname{Arg} z < \pi\}$, so that both $|\cdot|$ and Arg are continuous functions on D, with $|\cdot|$ taking values in (a, b), and Arg taking values in $(-\pi, \pi)$.

For each of $j = 0, 1$, write $\gamma_j(t) = r_j(t)e^{\theta_j(t)}$. That is,

$$r_j : [0,1] \to (a,b) \text{ by } r_j(t) = |\gamma_j(t)|, \quad \text{and} \quad \theta_j : [0,1] \to (-\pi,\pi) \text{ by } \theta_j(t) = \operatorname{Arg}(\gamma_j(t)),$$

both of which are continuous functions on $[0, 1]$, since each is a composition of continuous functions.

Define $R : [0,1] \times [0,1] \to (a,b)$ and $S : [0,1] \times [0,1] \to (-\pi,\pi)$ by

$$R(s,t) = (1-s)r_0(t) + sr_1(t) \quad \text{and} \quad S(s,t) = (1-s)\theta_0(t) + s\theta_1(t),$$

both of which are continuous functions, since each r_j and θ_j is continuous. Moreover, the image of R really lies in (a, b), since $r_j(t) \in (a, b)$ for all $t \in [0, 1]$ for each of $j = 0, 1$. Similarly, the image of S really lies in $(-\pi, \pi)$.

Thus, we may define $T : [0,1] \times [0,1] \to D$ by $\boxed{T(s,t) = R(s,t)e^{iS(s,t)}}$

The image of T really lies in D, since the images of R and S lie in (a, b) and $(-\pi, \pi)$, respectively. In addition, T is continuous since R and S are both continuous.

It remains to check that T has the correct boundary values. For any $s \in [0, 1]$, we have

$$R(s,0) = (1-s)|\gamma_0(0)| + s|\gamma_1(0)| = (1-s)|A| + s|A| = |A|,$$

and similarly $S(s, 0) = \operatorname{Arg}(A)$, $R(s, 1) = |B|$, and $S(s, 1) = \operatorname{Arg}(B)$.
Thus, $T(s, 0) = |A|e^{i \operatorname{Arg}(A)} = A$ and $T(s, 1) = |B|e^{i \operatorname{Arg}(B)} = B$ for all $s \in [0, 1]$.

Finally, for any $t \in [0, 1]$, we have $R(0, t) = r_0(t)$ and $S(0, t) = \theta_0(t)$, so that
$T(0, t) = r_0(t)e^{\theta_0(t)} = \gamma_0(t)$. Similarly, $T(1, t) = \gamma_1(t)$.

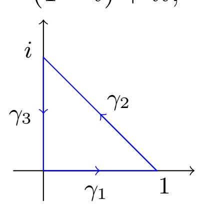
Thus, T is a homotopy fitting the required conditions. QED

Problem 2. IV.1, #1(a,b,c). Let γ be the boundary of the triangle $\{0 < x < 1, 0 < y < 1 - x\}$, oriented counterclockwise. Parametrize γ and use your parametrization to compute each of the following integrals:

(a) $\displaystyle\int_\gamma \operatorname{Re}(z)\, dz$ (b) $\displaystyle\int_\gamma \operatorname{Im}(z)\, dz$ (c) $\displaystyle\int_\gamma z\, dz$

Solutions. Define $\gamma_1, \gamma_2, \gamma_3 : [0, 1] \to \mathbb{C}$ by

$$\gamma_1(t) = t, \qquad \gamma_2(t) = (1-t) + it, \qquad \gamma_3(t) = i(1-t),$$



so that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Writing each γ_j as $x(t) + iy(t)$, with $dz = dx + idy$, this means

$$\text{for } \gamma_1: \qquad x = t,\ y = 0,\ dz = dt$$

$$\text{for } \gamma_2: \qquad x = 1-t,\ y = t,\ dz = (-1+i)dt$$
$$\text{for } \gamma_3: \qquad x = 0,\ y = 1-t,\ dz = -idt$$

Recalling $dz = dx + idy$, we can now compute the integrals:

(a): $\displaystyle\int_\gamma \mathrm{Re}(z)\,dz = \int_{\gamma_1} x\,dz + \int_{\gamma_2} x\,dz + \int_{\gamma_3} x\,dz$

$$= \int_0^1 t\,dt + \int_0^1 (1-t)(-1+i)\,dt + \int_0^1 0(-i)\,dt = \left[\frac{t^2}{2}\Big|_0^1\right] + (-1+i)\left[t - \frac{t^2}{2}\Big|_0^1\right] + 0$$

$$= \left(\frac{1}{2} - 0\right) + (-1+i)\left(1 - \frac{1}{2} - 0\right) = \boxed{\frac{i}{2}}$$

(b): $\displaystyle\int_\gamma \mathrm{Im}(z)\,dz = \int_{\gamma_1} y\,dz + \int_{\gamma_2} y\,dz + \int_{\gamma_3} y\,dz$

$$= \int_0^1 0\,dt + \int_0^1 t(-1+i)\,dt + \int_0^1 (1-t)(-i)\,dt = 0 + (-1+i)\left[\frac{t^2}{2}\Big|_0^1\right] + (-i)\left[t - \frac{t^2}{2}\Big|_0^1\right]$$

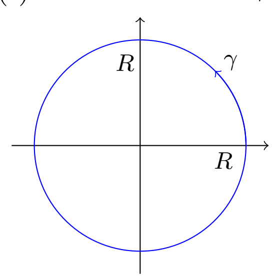
$$= 0 + (-1+i)\left(\frac{1}{2} - 0\right) + (-i)\left(1 - \frac{1}{2} - 0\right) = \boxed{-\frac{1}{2}}$$

(c): $\displaystyle\int_\gamma z\,dz = \int_\gamma \mathrm{Re}(z)\,dz + i\int_\gamma \mathrm{Im}(z)\,dz = \frac{i}{2} + i\left(-\frac{i}{2}\right) = \boxed{0}$

Problem 3. IV.1, #3, variant. Fix $R > 0$. Let γ be the circle $\{|z| = R\}$, oriented counterclockwise. Parametrize γ and use your parametrization to compute each of the following integrals for each integer $m = 0, \pm 1, \pm 2, \ldots$.

(a) $\displaystyle\int_\gamma z^m\,dz$ (b) $\displaystyle\int_\gamma |z|^m\,dz$ (c) $\displaystyle\int_\gamma \bar{z}^m\,dz$ (d) $\displaystyle\int_\gamma |z^m||\,dz|$

Solutions. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = Re^{it} = R\cos t + iR\sin t$:



Then since $x = R\cos t$ and $y = R\sin t$, we have $dx = -R\sin t\,dt$ and $dy = R\cos t\,dt$, so

$$dz = dx + i\,dy = R(-\sin t + i\cos t)\,dt = iR(\cos t + i\sin t)\,dt = iz\,dt$$

So we can now compute the integrals:

(a): $\displaystyle\int_\gamma z^m\,dz = \int_0^{2\pi} z^m(iz)\,dt = iR^{m+1}\int_0^{2\pi} e^{i(m+1)t}\,dt = iR^{m+1}\int_0^{2\pi} \cos\big((m+1)t\big) + i\sin\big((m+1)t\big)\,dt$

leading to two cases. If $m \neq -1$, then $m+1 \neq 0$, so

$$\int_\gamma z^m\,dz = \frac{iR^{m+1}}{m+1}\left(\sin\big((m+1)t\big) - i\cos\big((m+1)t\big)\Big|_0^{2\pi}\right) = \frac{iR^{m+1}}{m+1}\left((0-i) - (0-i)\Big|_0^{2\pi}\right) = 0.$$

Otherwise, we have $m = -1$, in which case $\displaystyle\int_\gamma z^m\,dz = i\int_0^{2\pi} e^0\,dt = i(2\pi - 0) = 2\pi i.$

That is, $\boxed{\displaystyle\int_\gamma z^m\,dz = \begin{cases} 2\pi i & \text{if } m=-1,\\ 0i & \text{if } m\neq -1.\end{cases}}$

(b): For $z = Re^it$ on the curve γ, we have $|z| = R$. Thus

$$\int_\gamma |z|^m\,dz = R^m\int_\gamma dz = R^m\int_\gamma z^0\,dz = R^m\cdot 0 = \boxed{0}$$

where we have applied part (a) with $m = 0$.

(c): For $z = Re^it$ on the curve γ, we have $\overline{z} = Re^{-it} = R^2z^{-1}$. Thus,

$$\int_\gamma \overline{z}^m\,dz = R^{2m}\int_\gamma z^{-m}\,dz = \boxed{\begin{cases} 2\pi iR^2 & \text{if } m=1,\\ 0i & \text{if } m\neq 1\end{cases}}$$

where we have applied part (a) with $-m$ in place of m.

(d): By our parametrization of γ, we have

$$|dz| = \sqrt{(x')^2+(y')^2}\,dt = \sqrt{R^2\sin^2 t + R^2\cos^2 t}\,dt = R\,dt$$

and $|z| = R$, so $\displaystyle\int_\gamma |z^m||\,dz| = \int_0^{2\pi} R^m\cdot R\,dt = \boxed{2\pi R^{m+1}}$

Problem 4. IV.1, #5. Use the ML-estimate to prove that $\displaystyle\left|\oint_{|z-1|=1}\frac{e^z}{z+1}\,dz\right| < 2\pi e^2.$

Proof. For all z on the path γ given by $|z-1| = 1$, we have $|\operatorname{Re} z - 1| \leq |z-1| = 1$, and hence $\operatorname{Re}(z)\leq 2$. Therefore, $|e^z| = e^{\operatorname{Re} z}\leq e^2$.

Also for z on γ, we have $2 = |(z+1)-(z-1)| \leq |z+1| + |z-1| = |z+1|+1$, and hence $|z+1|\geq 1$.

Thus, $\displaystyle\left|\frac{e^z}{z+1}\right| \leq \frac{e^2}{1} = e^2$. So use $M = e^2$.

The curve γ is a circle of radius 1, so its length is $L = 2\pi$.

Therefore, by the ML-estimate, $\displaystyle\left|\oint_{|z-1|=1}\frac{e^z}{z+1}\,dz\right| \leq ML = 2\pi e^2$. But we need $<$.

So split γ into the right half γ_1 and the left half γ_2. Both of these curves have length $L' = \pi$.

For any z on γ_1, we have $\operatorname{Re} z\geq 1$, so $|z+1|\geq|\operatorname{Re} z+1|\geq 2$. Thus, for any such z, we have $\displaystyle\left|\frac{e^z}{z+1}\right|\leq\frac{e^2}{2}$. So we may use $M_1 = \dfrac{e^2}{2}$ on γ_1, while keeping the same bound M on γ_2.

Thus, applying the bound M_1L' to γ_1 and ML' to γ_2, we have:

$$\left|\oint_{|z-1|=1}\frac{e^z}{z+1}\,dz\right| \leq \left|\oint_{\gamma_1}\frac{e^z}{z+1}\,dz\right| + \left|\oint_{\gamma_2}\frac{e^z}{z+1}\,dz\right| \leq \frac{e^2}{2}\cdot\pi + e^2\cdot\pi = \frac{3}{2}e^2\pi < 2\pi e^2 \qquad \text{QED}$$

Note 1: The key inequalities $\operatorname{Re} z\leq 2$ and $|z+1|\geq 1$ can also be seen just by drawing a picture of the circle $|z-1| = 1$, with no careful inequality computations. The largest real part on γ is at the rightmost point of the circle, at $z = 2$, so $\operatorname{Re} z\leq 2$ for all z on γ. And the closest point to -1 is at $z = 0$, so $|z+1|\geq 1$ for all z on γ.

Note 2: One can also reduce the bound on γ_2 to $M_2 = e$, since $\operatorname{Re} z\leq 1$ there, so $|e^z|\leq e$. But we only need to prove a slight improvement on the $2\pi e^2$ bound.