Math 345, Fall 2024

Solutions to Homework #10

Problem 1. III.1, #1(a,c). Compute $\int_\gamma y^2\,dx + x^2\,dy$ along each of the following two paths γ from $(0,0)$ to $(2,4)$:

- γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.
- γ is the vertical line segment $(0,0)$ to $(0,4)$, followed by the horizontal line segment from $(0,4)$ to $(2,4)$.

Solution. First path: parametrize by $x(t) = t$, $y(t) = t^2$ for $0 \le t \le 2$.
Then $dx = dt$ and $dy = 2t\,dt$. So the integral is

$$\int_0^2 (t^2)^2 + t^2(2t)\,dt = \int_0^2 t^4 + 2t^3\,dt = \frac{1}{5}t^5 + \frac{1}{2}t^4\Big|_0^2 = \frac{32}{5} + \frac{16}{2} - 0 = \frac{32+40}{5} = \boxed{\frac{72}{5}}$$

Second path: First leg: parametrize by $x(t) = 0$, $y(t) = t$ for $0 \le t \le 4$.
Then $dx = 0$ and $dy = dt$. So the integral is $\int_0^4 0 + 0\,dt = 0$

Second leg: parametrize by $x(t) = t, y(t) = 4$ for $0 \le t \le 2$.
Then $dx = dt$ and $dy = 0$. So the integral is $\int_0^2 4^2 + 0\,dt = 16t\Big|_0^2 = 32$.

So the full integral is $0 + 32 = \boxed{32}$

Problem 2. III.1, #4. Fix $R > 0$, and let γ be the semicircle in the upper half-plane from R to $-R$. Evaluate $\int_\gamma y\,dx$ in two ways: first directly, and then using Green's Theorem.

Solution. First method: parametrize γ by $x(t) = R\cos t$, $y(t) = R\sin t$ for $0 \le t \le \pi$.
Then $dx = -R\sin t$, so the integral is $\int_0^\pi -R^2\sin^2 t\,dt = \frac{-R^2}{2}\int_0^\pi 1 - \cos 2t\,dt$

$$= \frac{-R^2}{2}\left(t - \frac{1}{2}\sin 2t\right) = \frac{-R^2}{2}\big((\pi - 0) - (0 - 0)\big) = \boxed{-\frac{\pi}{2}R^2}$$

Second method: let γ' be the path from $-R$ to R parametrized by $x = t$, $y = 0$ for $-R \le t \le R$.
Then $dx = dt$, so $\int_{\gamma'} y\,dx = \int_0^\pi 0\,dt = 0$.

Let D be the half-disk enclosed by $\gamma + \gamma'$. Let $P = y$ and $Q = 0$, so that $\frac{\partial Q}{x} - \frac{\partial P}{y} = 0 - 1 = -1$.
Then by Green's Theorem (noting that $\gamma + \gamma'$ is oriented positively with respect to D), we have

$$\int_\gamma y\,dx = \int_\gamma y\,dx + 0 = \int_\gamma y\,dx + \int_{\gamma'} y\,dx = \int_{\partial D} y\,dx = \iint_D -1\,dA = -\text{Area}(D) = \boxed{-\frac{\pi}{2}R^2}$$

[*Note*: of course, one can instead compute $\iint_D -1\,dA$ by hand. Using polar coordinates, it is $\int_0^\pi\int_0^R (-1)r\,dr\,d\theta$, which soon evaluates to $-\frac{\pi}{2}R^2$ as above.]

Problem 3. III.1, #5, variant. Let D be a bounded region in the plane with piecewise smooth boundary curve ∂D. Use Green's Theorem to prove that $\int_{\partial D} x\,dy$ is the area of D.

Proof. Let $P = 0$ and $Q = x$, so that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - 0 = 1$.

By Green's Theorem, then: $\displaystyle\int_{\partial D} x\,dy = \int_{\partial D} P\,dx + Q\,dy = \iint_D 1\,dA = \text{Area}(D)$ QED

Problem 4. III.2, #1(b,c,d). For each of the following differential forms ω, determine whether ω is independent of path or not.

If yes, find a function h such that $dh = \omega$. If no, find a closed path γ around the origin such that $\displaystyle\int_\gamma \omega \neq 0$.

(b) $\omega = x^2\,dx + y^5\,dy$ (c) $\omega = y\,dx + x\,dy$ (d) $\omega = y\,dx - x\,dy$

Solutions. (b): $\boxed{\text{Yes, independent of path}}$

We need $h_x = x^2$ and $h_y = y^5$, so we may choose $\boxed{h = \dfrac{x^3}{3} + \dfrac{y^6}{6}}$ which has $dh = \omega$.

(c): $\boxed{\text{Yes, independent of path}}$

We need $h_x = y$ and $h_y = x$. The first condition gives $h = xy + g(y)$, so $h_y = x + g'(y)$. Therefore we need $g'(y) = 0$, so we may choose $g = 0$. That is, we may choose $\boxed{h = xy}$ which has $dh = \omega$.

(d): $\boxed{\text{No, not independent of path}}$ since $Q_x - P_y = -1 - 1 = -2 \neq 0$.

Let γ be the (counterclockwise) circular path $|z| = 1$, which we parametrize by $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$. Then $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, so

$$\int_\gamma \omega = \int_0^{2\pi} (\sin t)(-\sin t) + (-\cos t)(\cos t)\,dt = \int_0^{2\pi} (-1)\,dt = -t\Big|_0^{2\pi} = -2\pi + 0 = -2\pi \neq 0$$

Problem 5. III.2, #3. Fix $b > a > 0$, and let D be the annulus $a < |z| < b$. Let $P, Q : D \to \mathbb{R}$ be smooth functions such that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. Use Green's Theorem to prove that the value of $\displaystyle\oint_{|z|=r} P\,dx + Q\,dy$ is independent of the radius r, for $a < r < b$.

Proof. Given arbitrary $r, s \in (a, b)$, we must show that $\displaystyle\oint_{|z|=r} P\,dx + Q\,dy = \oint_{|z|=s} P\,dx + Q\,dy$.

If $r = s$, this is clearly true, and so without loss of generality, we may assume that $s > r$.

Let E be the open annulus $r < |z| < s$, which is contained in D. Note that the boundary of E consists of the circle $|z| = s$ traced counterclockwise together with the circle $|z| = r$ traced clockwise. Thus,

$$\oint_{|z|=s} P\,dx + Q\,dy - \oint_{|z|=r} P\,dx + Q\,dy = \int_{\partial E} P\,dx + Q\,dy = \iint_E \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,dA = \iint_E 0\,dA = 0,$$

where the second equality is by Green's Theorem and the third is by hypothesis.

Adding $\displaystyle\oint_{|z|=r} P\,dx + Q\,dy$ to both sides, then we have $\displaystyle\oint_{|z|=r} P\,dx + Q\,dy = \oint_{|z|=s} P\,dx + Q\,dy$. QED