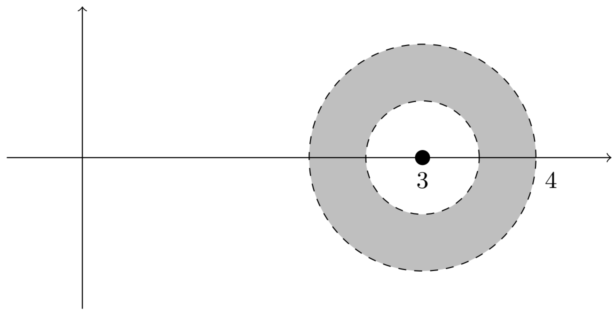
Math 345, Fall 2024

Solutions to Selected Homework Problems, HW #1

I.1, #1(b,e): Identify and sketch these sets of points. (b): $1 < |2z - 6| < 2$ and (e) $|z - 1| < |z|$.

Solutions. (b): Since $|2z - 6| = |2| \cdot |z - 3| = 2|z - 3|$, this set is given by $\frac{1}{2} < |z - 3| < 1$. That is, it is the set of points $z \in \mathbb{C}$ whose distance from 3 is strictly between 1/2 and 1. So it's this open annulus:



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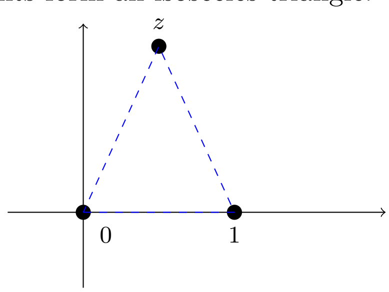
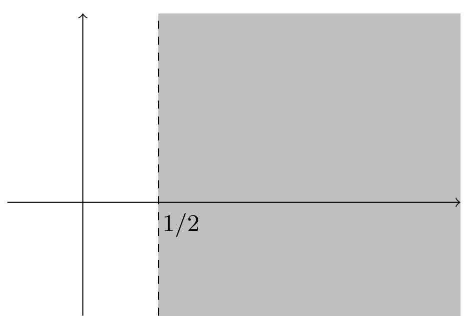
(e): Since $|z-1|$ is the distance from z to 1, and $|z|$ is the distance from z to 0, the set where $|z-1| < |z|$ is the set of points $z \in \mathbb{C}$ which are closer to 1 than to 0. The set of points where these two distances are the same is the vertical line $\operatorname{Re} z = \frac{1}{2}$, because on this line, the distances from z to each of 0 and 1 are the same, since the three points form an isosceles triangle:



So the desired set is the set of points strictly to the *right* of that vertical line, which is this open half-plane:



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Alternative argument in (c) to see that the region is $\operatorname{Re} z > 1/2$. Squaring both sides of the (real) inequality $|z - 1| < |z|$ gives $|z - 1|^2 < |z|^2$. Writing $z = x + iy$, this inequality is just $(x - 1)^2 + y^2 < x^2 + y^2$, which expands to $-2x + 1 < 0$, i.e., $1 < 2x$, or equivalently $x > 1/2$. That is, $\operatorname{Re} z > 1/2$.

I.1, #2(a): Verify the identity $\overline{zw} = \overline{z}\,\overline{w}$

Proof. Given $z, w \in \mathbb{C}$, write $z = x + iy$ and $w = u + iv$ with $x, y, u, v \in \mathbb{R}$. Then

$\overline{zw} = \overline{(xu - yv) + i(xv + yu)} = (xu - yv) - i(xv + yu) = (x - iy)(u - iv) = \overline{z}\,\overline{w}$ QED

I.1, #5: Show that (for all $z, w \in \mathbb{C}$) $|\operatorname{Re} z| \le |z|$, that $|\operatorname{Im} z| \le |z|$, and that $|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$. Use this to prove the triangle inequality.

Proof. For the first two statements, given $z \in \mathbb{C}$, write $z = x + iy$. Then
$|\operatorname{Re} z| = |x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = |z|$ and $|\operatorname{Im} z| = |y| = \sqrt{y^2} \le \sqrt{x^2 + y^2} = |z|$.

For the third statement, given $z, w \in \mathbb{C}$, let $c = z\overline{w} \in \mathbb{C}$. Then
$|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w = |z|^2 + |w|^2 + c + \overline{c}$,
since $\overline{c} = \overline{z\overline{w}} = \overline{z}\overline{\overline{w}} = \overline{z}w$.
Writing $c = a + bi$ with $a, b \in i$, we have $c + \overline{c} = (a + bi) + (a - bi) = 2a = 2\operatorname{Re}(c)$.
Thus, $|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(c) = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$.

Finally, to prove the triangle inequality, the previous identity gives

$$|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}) \le |z|^2 + |w|^2 + 2|\operatorname{Re}(z\overline{w})| \le |z|^2 + |w|^2 + 2|z\overline{w}|$$
$$= |z|^2 + |w|^2 + 2|z||\overline{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.$$

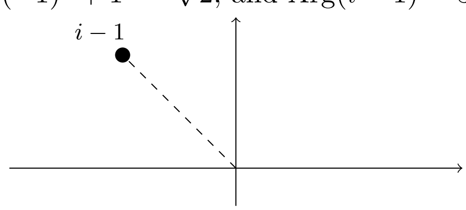
All quantities here are nonnegative real numbers, so taking square roots gives $|z + w| \le |z| + |w|$. QED

Alternative proof of third identity. Given $z, w \in \mathbb{C}$, write $z = x + iy$ and $w = u + iv$ with $x, y, u, v \in \mathbb{R}$. Then

$$|z + w|^2 = |(x + u) + i(y + v)|^2 = (x + u)^2 + (y + v)^2 = x^2 + 2xu + u^2 + y^2 + 2yv + v^2$$
$$= (x^2 + y^2) + (u^2 + v^2) + 2(xu + yv) = |z|^2 + |w|^2 + 2\operatorname{Re}\big((xu + yv) + i(yu - xv)\big)$$
$$= |z|^2 + |w|^2 + 2\operatorname{Re}\big((x + iy)(u - iv)\big) = |z|^2 + |w|^2 + 2\operatorname{Re}\big(z\overline{w}\big)$$

I.2, #1(b): Find all values of $\sqrt{i - 1}$ in both polar and cartesian, and plot them.

Solution. We have $|i - 1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, and $\operatorname{Arg}(i - 1) = 3\pi/4$, as we see here:



Thus, the two square roots of $i - 1$ are of the form $re^{i\theta}$ where $r = \sqrt[4]{2}$ and $\theta = \dfrac{3}{8}\pi + \dfrac{2\pi}{2}n$ for $n = 0, 1$.
[To make the numbers smaller, let's actually subtract 2π from the second θ to get $-5\pi/8$.]
So in polar, the two roots are $\boxed{\sqrt[4]{2}e^{3\pi i/8} \text{ and } \sqrt[4]{2}e^{-5\pi i/8}}$
which in cartesian are $\boxed{\sqrt[4]{2}\cos(3\pi/8) + i\sqrt[4]{2}\sin(3\pi/8)}$ and $\boxed{\sqrt[4]{2}\cos(5\pi/8) - i\sqrt[4]{2}\sin(5\pi/8)}$

Note 1: When I wrote the cartesian answers above, I used the facts that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Note 2: the second root is, of course, simply the negative of the first, since it is $e^{i\pi}$ times the first.

Note 3: You could "simplify" those cartesian formulas a bit using the half-angle identities to get $\cos^2(3\pi/8) = \frac{1}{2}(1 + \cos(3\pi/4)) = \frac{\sqrt{2}-1}{2\sqrt{2}}$ and hence $\cos(3\pi/8) = \frac{\sqrt{\sqrt{2}-1}}{2^{3/4}}$. Similarly $\sin(3\pi/8) = \frac{\sqrt{\sqrt{2}+1}}{2^{3/4}}$.

So the cartesian answers would be $\pm\left(\dfrac{\sqrt{\sqrt{2} - 1}}{\sqrt{2}} + i\dfrac{\sqrt{\sqrt{2} + 1}}{\sqrt{2}}\right)$, but I'm not sure that really looks any nicer.

I.2, #4: For which integers $n \geq 1$ is i an n-th root of unity?

Solution/Proof. We are being asked for the set of integers $n \geq 1$ for which $i^n = 1$. We claim that this set is precisely $\boxed{\text{those integers } n \geq 1 \text{ that are divisble by } 4}$

To see this, writing an arbitrary integer $n \geq 1$ as $n = 4k + j$, where k is an integer and $j \in \{0, 1, 2, 3\}$. We must show that $i^n = 1$ if and only if $j = 0$.
If $j = 0$, then $i^n = i^{4k} = (i^4)^k = 1^k = 1$, as desired.
Conversely, if $j \in \{1, 2, 3\}$, then $i^n = i^{4k} \cdot i^j = i^j \neq 1$, since $i^1 = i$ and $i^2 = -1$ and $i^3 = -i$. QED

I.2, #8: Prove that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\cos\theta\sin\theta$ using DeMoivre's formulae. Find corresponding formulae for $\cos(4\theta)$ and $\sin(4\theta)$.

Solution/Proof. DeMoivre for $n = 2$ says, for all $\theta \in \mathbb{R}$

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

Taking real parts of both sides gives $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.
Taking imaginary parts gives $\sin(2\theta) = 2\cos\theta\sin\theta$, as desired.

For $n = 4$, DeMoivre gives

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Taking real parts gives $\boxed{\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Taking imaginary parts gives $\boxed{\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

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Alternative derivation of last identities. Applying the 2θ identities to 2θ itself, we have

$$\cos(4\theta) = \cos\big(2(2\theta)\big) = \cos^2(2\theta) - \sin^2(2\theta) = \big(\cos^2(\theta) - \sin^2(\theta)\big)^2 - \big(2\cos\theta\sin\theta\big)^2$$
$$= \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta - 4\cos^2\theta\sin^2\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and $\sin(4\theta) = \sin\big(2(2\theta)\big) = 2\cos(2\theta)\sin(2\theta) = 2\big(\cos^2(\theta) - \sin^2(\theta)\big) \cdot \big(2\cos\theta\sin\theta\big)$

$$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$