Math 345, Fall 2024 Professor Rob Benedetto

Practice Problems for Midterm Exam 2

1. Compute the following integrals.

(a) $\displaystyle\int_{|z|=6} \frac{\cos(\pi z)}{(z-1)^3}\,dz$ (b) $\displaystyle\int_{|z|=2} \frac{e^{i\pi z/2}}{z(z+1)(z+3)}\,dz$

(c) $\displaystyle\int_{|z-3|=2} \frac{\operatorname{Log} z}{z^2(z-4)^2}\,dz$ (d) $\displaystyle\int_{|z|=4} \frac{e^{5z}}{(z-\pi i)^3}\,dz$

(e) $\displaystyle\int_{|z|=\pi} \frac{e^{5z}}{(z-4)^4}\,dz$ (f) $\displaystyle\int_{|z-5|=4} \frac{(z-3)\sin z}{z^3(z-6)(z-8)}\,dz$

2. For each of the following functions, find its full power series expansion about $z=0$, as well as the radius of convergence of this power series.

(a) $f(z) = z\operatorname{Log}(z+2)$ (b) $g(z) = \dfrac{z^2}{(z^5-4)^3}$

3. Let $h(z) = (z^2+1)\sin(2z^3)$. Compute the following derivatives: $h^{(15)}(0)$, $h^{(16)}(0)$, and $h^{(17)}(0)$. [You do not need to simplify or expand expressions like $5^7 \cdot (17!)/(25!)$.]

4. (a) Find the power series centered at $z=0$ of these two functions:

$$g(z) = ze^{(z^2)} \qquad \text{and} \qquad h(z) = \cos(2z).$$

(b) Consider the power series $\displaystyle\sum_{k=0}^{\infty} a_k z^k$ centered at $z=0$ for $f(z) = \dfrac{ze^{(z^2)}}{\cos(2z)}$.
Use part (a) to compute a_k for each of $k = 0, \ldots, 6$.

(c) For $f(z)$ as in part (b), compute $f^{(5)}(0)$.

(d) What is the radius of convergence of the power series in part (b)?
(Briefly) explain why.

5. Let $D = \{z \in \mathbb{C} : \operatorname{Re} z \le -2\}$. Prove that $\displaystyle\sum_{n=1}^{\infty} n^z$ converges uniformly on D, where n^z denotes $e^{z \operatorname{Log} n}$.

6. Let $E = \{z \in \mathbb{C} : |z| \ge 7\}$. Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{5k - z^{3k}}$ converges uniformly on E.

7. Let f be an entire function with the property that for all $z \in \mathbb{C}$,

$$f(z+1) = f(z+i) = f(z).$$

Prove that f is constant.