**Path-Independence for Homotopic Paths**

This handout presents a rigorous proof of the following theorem, which appears in Section III.2 of Gamelin's text, at the top of page 81:

**Theorem**. Let $D \subseteq \mathbb{R}^2$ be a domain, let $A, B \in D$, let $\gamma_0$ and $\gamma_1$ be paths from $A$ to $B$ in $D$, and let $P\,dx + Q\,dy$ be a (smooth) closed differential form in $D$. If $\gamma_0$ is homotopic to $\gamma_1$, then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy.$$

---

Recall the following definitions:

We say that $P\,dx + Q\,dy$ is **closed** if:

$P$ and $Q$ are $C^1$ (i.e., they have continuous first partial derivatives), and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

If $\gamma_0, \gamma_1 : [a,b] \to D$ are paths from $A$ to $B$ in $D$, we say that $\gamma_0$ is **homotopic** to $\gamma_1$ if there is a *continuous* function $T : [0,1] \times [a,b] \to D$ such that

- $T(0,t) = \gamma_0(t)$ for all $t \in [a,b]$,
- $T(1,t) = \gamma_1(t)$ for all $t \in [a,b]$,
- $T(s,a) = A$ for all $s \in [0,1]$, and
- $T(s,b) = B$ for all $s \in [0,1]$.

Note that Gamelin writes $\gamma_s(t)$ instead of $T(s,t)$, to emphasize that intuitively, we think of $T$ as a continuously varying family of paths from $A$ to $B$.

That is, for each $s \in [0,1]$, the function $\gamma_s : [a,b] \to D$ is a path from $A$ to $B$ in $D$, and $\gamma_s$ is close to $\gamma_r$ whenever $s$ is close to $r$.

---

**Proof of Theorem**. Define $K = [0,1] \times [a,b]$, which is a closed rectangle. Then $K$ is compact, since it is a closed and bounded subset of $\mathbb{R}^2$. We will cover $K$ by open disks as follows:

For each point $(s,t) \in K$, because $T(s,t) \in D$ and $D$ is open, $\boxed{\text{there is some } \varepsilon > 0}$ so that we have $D\big(T(s,t), \varepsilon\big) \subseteq D$. Since $T$ is continuous, then, $\boxed{\text{there is some } \delta' > 0}$ such that

$$T\Big(D\big((s,t), \delta'\big) \cap K\Big) \subseteq D\big(T(s,t), \varepsilon\big).$$

[That is, for every point $(x,y) \in K$ with $\|(x,y) - (s,t)\| < \delta'$, we have $\|T(x,y) - T(s,t)\| < \varepsilon$.]

$\boxed{\text{Define } \delta_{s,t} > 0 \text{ to be } \delta'/3}$ and observe that the disk $D\big((s,t), \delta_{s,t}\big)$ contains $(s,t)$.

Thus, we have a covering $\{D((s,t), \delta_{s,t})\}_{(s,t) \in K}$ of $K$ by open disks. (One disk for each of the infinitely many points of $K$!) But **since** $K$ **is compact**, there is a finite subcover

$$\Big\{D\big((s_1,t_1), \delta_1\big), D\big((s_2,t_2), \delta_2\big), \ldots, D\big((s_N,t_N), \delta_N\big)\Big\}.$$

That is, for every $(s,t) \in K$, there is some $j \in \{1, \ldots, N\}$ such that $(s,t) \in D((s_j,t_j), \delta_j)$.

$\boxed{\text{Define } \delta = \min\{\delta_1, \ldots, \delta_N\} > 0}$

---

**Side Note**: Why did we do that crazy open covering above? The answer is that we wanted a **single** number $\delta > 0$ that has a nice property at **every** point $(s,t)$ in the original rectangle $K$. The problem is that there are infinitely many points $(s,t) \in K$, so we can't just take the minimum (or really, infimum) of all of the radii $\delta_{s,t}$, since the infimum of infinitely many positive numbers might be 0.

So we needed to restrict ourselves to only **finitely many** points $(s_1,t_1), \ldots, (s_N,t_N)$, so that we could take the minimum of the corresponding finitely many $\delta_j$'s. The infimum of infinitely many positive $\delta_j$'s might be zero; but the infimum (i.e., minimum) of finitely many positive $\delta_j$'s is **still positive**.

---

OK, back to the proof. We now have a claim to make about this $\delta$ we just made:

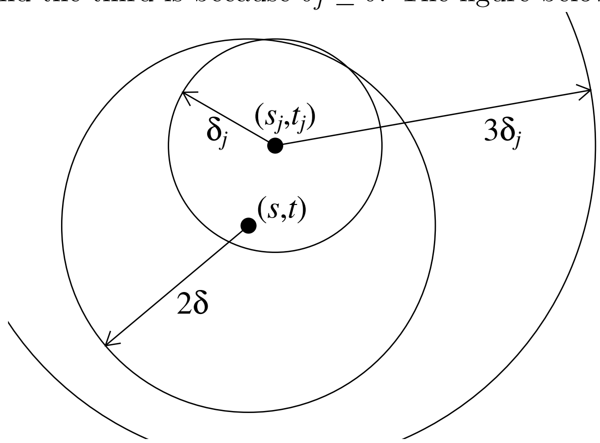
**Claim**. For each point $(s,t) \in K$, there is some $j \in \{1,\dots,N\}$ such that

$$D\big((s,t),2\delta\big) \subseteq D\big((s_j,t_j),3\delta_j\big).$$

**Proof of Claim**. Denote $\|(x,y)\| = \sqrt{x^2+y^2}$, so that the distance between $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ is $\big\|(x_1,y_1)-(x_2,y_2)\big\|$.
We may pick $j \in \{1,\dots,N\}$ such that $(s,t) \in D((s_j,t_j),\delta_j)$. Given a point $(s',t') \in D((s,t),2\delta)$, we have

$$\big\|(s',t')-(s_j,t_j)\big\| \leq \big\|(s',t')-(s,t)\big\| + \big\|(s,t)-(s_j,t_j)\big\| < 2\delta + \delta_j \leq 3\delta_j,$$

where the first inequality is the triangle inequality, the second is because $(s',t') \in D((s,t),2\delta)$ and $(s,t) \in D((s_j,t_j),\delta_j)$, and the third is because $\delta_j \geq \delta$. The figure below may help explain this:



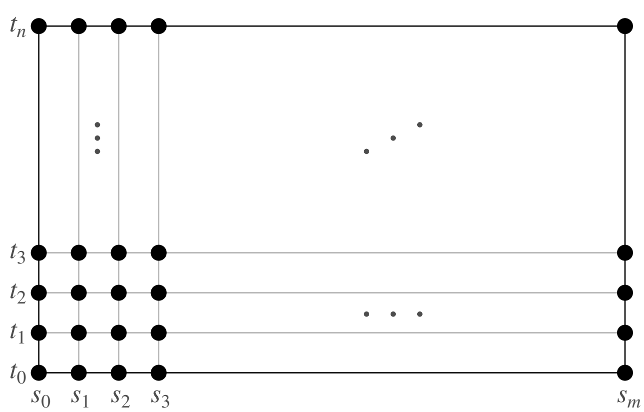

QED Claim

[**Note**: the Claim, together with the way each $\delta_j$ was chosen earlier — as $\delta'/3$, rather than just as $\delta'$ — shows that for *any* point $(s,t) \in K$, the image of $D((s,t),2\delta) \cap K$ under $T$ is completely contained in a disk $D(P,\varepsilon)$ that is itself contained in $D$.]

Continuing with the proof of the theorem: Pick real numbers $s_0,s_1,\dots,s_m$ and $t_0,t_1,\dots,t_n$ so that

$$0 = s_0 < s_1 < s_2 < \cdots < s_m = 1, \quad \text{and} \quad a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

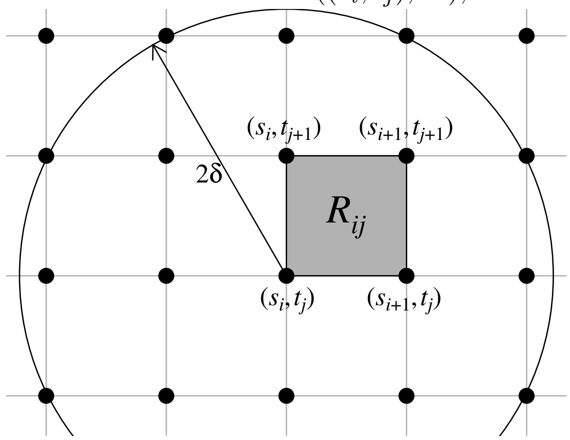
and also so that $s_{i+1} - s_i < \delta$ and $t_{j+1} - t_j < \delta$ for each $i = 0,\dots,m-1$ and $j = 0,\dots,n-1$. [For example, pick $m,n \geq 1$ big enough that $\Delta s := 1/m < \delta$ and $\Delta t := (b-a)/n < \delta$, and then define $s_j := j\Delta s$ and $t_j := a + j\Delta t$ for each subscript $j$.] The points $(s_i,t_j)$ appear as the (big) dots in the rectangle $K$ as in the figure below:



Note that for each $i \in \{0, 1, \dots, m-1\}$ and $j \in \{0, 1, \dots, n-1\}$, every point $(s,t)$ in the rectangle $R_{ij}$ with corners at $(s_i, t_j)$, $(s_i, t_{j+1})$, $(s_{i+1}, t_{j+1})$, and $(s_{i+1}, t_j)$ is distance less than $2\delta$ from $(s_i, t_j)$. After all, the distance in question is

$$\sqrt{(s-s_i)^2 + (t-t_j)^2} \le \sqrt{(s_{i+1}-s_i)^2 + (t_{j+1}-t_j)^2} < \sqrt{2\delta^2} < 2\delta.$$

That is, the rectangle $R_{ij}$ is contained in the disk $D((s_i, t_j), 2\delta)$, as in the following diagram:




By the Claim and the previous paragraph, then, the image $T(R_{ij})$ of the $(i,j)$-th rectangle is contained in an open disk $D_{ij} = D(T(s_i, t_j), \varepsilon)$ contained in $D$. Since $D_{ij}$ is convex and hence star-shaped, our theorem for star-shaped domains says that

$$\int_{T(\partial R_{ij})} P\,dx + Q\,dy = 0,$$

since $T(\partial R_{ij})$ is a path in $D_{ij}$ starting and ending at the same point, and $P\,dx + Q\,dy$ is closed. Summing this equality over all the rectangles $R_{ij}$ for $0 \le i \le m-1$ and $0 \le j \le n-1$, we get

$$\int_{T(\partial K)} P\,dx + Q\,dy = 0,$$

since for any interior edge $E$, say from $(s_i, t_j)$ to $(s_{i+1}, t_j)$ for $1 \le j \le n-1$, the path $T(E)$ is traced in one direction in the integral over $T(\partial R_{ij})$ and in the opposite direction in the integral over $T(\partial R_{i(j-1)})$. That is, only images of the exterior edges $T(\partial K)$ survive cancellation in the sum.

Explicitly, however, $T(\partial K)$ consists of four paths, one along each of the four edges of the original rectangle $K = [0,1] \times [a,b]$. But the assumption that $T$ is a homotopy says precisely what $T$ does on each of these edges!

First, the image $T(E_1)$ of the bottom edge $E_1 = [0,1] \times \{a\}$ is simply the constant path at $A$.

Similarly, the image $T(E_3)$ of the top edge $E_3 = [0,1] \times \{b\}$ (traced backwards) is the constant path at $B$.

Next, the image $T(E_2)$ of the right edge $E_2 = \{1\} \times [a,b]$ is the path $\gamma_1$.

Finrally, the image $T(E_4)$ of the left edge $E_4 = \{0\} \times [a,b]$ is the path $\gamma_0$, but traced backwards.

Thus, the integral around $\partial K$ becomes simply

$$\int_{\gamma_1} P\,dx + Q\,dy - \int_{\gamma_0} P\,dx + Q\,dy = 0,$$

from which the Theorem follows immediately. QED