Math 345, Fall 2024 Professor Rob Benedetto

Homework #7

Due **Monday, September 30** in Gradescope by **11:59 pm ET**

- **WATCH** Video 8: Proof of the Chain Rule
- **READ** Sections II.4 and II.5 of Gamelin, up to the halfway point of page 55
- **READ** The discussion below about Jacobians, which is sort of from Math 211
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Definition. Let $R, S \subseteq \mathbb{R}^2$ be open sets in the plane, and let $T : R \to S$ given by $T(x, y) = \big(g(x, y), h(x, y)\big)$. Suppose that both g and h are differentiable on T. The **Jacobian** of T at any point $(x, y) \in R$ is the 2×2 matrix $J_T = \begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix} = \begin{bmatrix} g_x(x,y) & g_y(x,y) \\ h_x(x,y) & h_y(x,y) \end{bmatrix}$.

Theorem (Multivariable Change-of-Variables Formula).
Let $R, S \subseteq \mathbb{R}^2$ be open sets, and let $T : R \to S$ given by $T(x, y) = \big(g(x, y), h(x, y)\big)$ be continuously differentiable. (I.e., all four partial derivatives g_x, g_y, h_x, h_y are continuous on R.) Suppose further that T is one-to-one and onto. Let $F : R \to \mathbb{R}$ be continuous. Then

$$\iint_S F(u, v)\, du\, dv = \iint_R F\big(T(x, y)\big)\big|\det(J_T)\big|\, dx\, dy.$$

(Note: that's the absolute value of the determinant of the Jacobian, i.e. $|g_x h_y - g_y h_x|$, appearing in the second integral.)

Theorem (The Inverse Function Theorem).
Let $R \subseteq \mathbb{R}^2$ be an open set in the plane, and let $T : R \to \mathbb{R}^2$ be a continuously differentiable function given by $T(x, y) = \big(g(x, y), h(x, y)\big)$. Let $(x_0, y_0) \in R$ be a point such that $\det(J_T(x_0, y_0)) \neq 0$. [That is, $J_T(x_0, y_0)$ is an invertible matrix.]

Then there is some $\varepsilon > 0$ such that:

- the open disk $U = D((x_0, y_0), \varepsilon)$ is contained in R,
- T is one-to-one on U,
- the set $V = T(U) \subseteq \mathbb{R}^2$ is also open,
- the function $T^{-1} : V \to U$ is continuously differentiable, and
- the Jacobian $J_{T^{-1}}$ of T^{-1} satisfies $J_{T^{-1}}\big(T(x, y)\big) = \big[J_T(x, y)\big]^{-1}$ for all $(x, y) \in U$.

Note that the second bulleted conclusion, together with the definition of V as $T(U)$, means that $T : U \to V$ is one-to-one and onto. Also note that the final bullet is the main conclusion, and it says that the Jacobian of the inverse function T^{-1} is the (matrix) inverse of the Jacobian of the original function T. That formula can be equivalently written as: $J_{T^{-1}}(u, v) = \big[J_T\big(T^{-1}(u, v)\big)\big]^{-1}$ for all $(u, v) \in V$.

Next, complete the HW problems

found on the next page

Assigned Problems for HW 7

Problem 1. II.4, #2. Let $a \in \mathbb{C} \setminus \{0\}$ be a nonzero complex number, and let $f(z)$ be an analytic branch of z^a . Prove that $f'(z) = af(z)/z$.

Problem 2. II.4, #7. Let $D \subseteq \mathbb{C}$ be a bounded domain, and let f be a bounded analytic function on D. Suppose also that f is one-to-one on D. Prove that

$$\text{Area}\big(f(D)\big) = \iint_D \left|f'(z)\right|^2 dx\, dy.$$

[*Hint*: Use the Multivariable Change-of-Variables Formula and facts from Section II.4.]

Problem 3. II.4, #9. Let $D = D(0,1)$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and let $f(z) = z^2$. Compute $\iint_D \left|f'(z)\right|^2 dx\, dy$. Interpret the answer in terms of areas; that is, explain how the value you get actually agrees with the previous problem.

Problem 4. II.5, #2. Let $D \subseteq \mathbb{C}$ be a domain, and let $u, v : D \to \mathbb{R}$ be harmonic functions. Suppose that v is a harmonic conjugate for u. Prove that $-u$ is a harmonic conjugate for v.

Problem 5. II.5, #3(a,b), slight variant. Define $u(z) = \begin{cases} \text{Im}(1/z^2) & \text{for } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{for } z = 0. \end{cases}$

Prove that all four of $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial y^2}$ exist at *all* points of $\mathbb{C}$ (viewed as $\mathbb{R}^2$), including at the point $(0,0)$. Then verify that u satisfies Laplace's equation on $\mathbb{R}^2$. That is, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ everywhere on $\mathbb{R}^2$.

[*Note*: Most of the work required here is about proving it at $(0,0)$. On $\mathbb{C} \setminus \{0\}$, you can compute the partial derivatives by hand if you really want to, but it's much faster to invoke facts about analytic functions.]

Problem 6. II.5, #3(c,d), slight variant. With u as in the previous problem, prove that $\dfrac{\partial^2 u}{\partial x \partial y}$ does *not* exist at $(0,0)$. Conclude that u is *not* harmonic on the whole plane, even though we just saw that it satisfies Laplace's equation on the whole plane.

Optional Challenges:

A: II.4, #5: Use the formula $\tan^{-1} z = \dfrac{1}{2i} \log\left(\dfrac{1+iz}{1-iz}\right)$ to find the derivative first of the principal branch (using Log) and then of any branch of $\tan^{-1}(z)$.

B: II.4, #5: Find the derivative(s) of any branch $g(z)$ of $\cos^{-1}(z) = -i \log\left[z \pm \sqrt{z^1 - 1}\right]$.