Math 345, Fall 2024 Professor Rob Benedetto

Homework #6

Due **Thursday, September 26** in Gradescope by **11:59 pm ET**

- **WATCH** Video 7: Compact Sets
- **READ** Sections II.2 and II.3 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. II.1, #16, slight variant: Prove that
(a) the slit plane $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped, but
(b) the punctured plane $\mathbb{C} \setminus \{0\}$ is *not* star-shaped.

Problem 2. II.2, #2. For any $z \in \mathbb{C} \setminus \{1\}$ and any integer $n \geq 1$, prove that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - z^n}{(1-z)^2} - \frac{nz^n}{1-z}.$$

Problem 3. II.2, #3. Prove (from the definition) that the functions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are not differentiable at any point.

Problem 4. II.3, #2. Prove that the functions $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the Cauchy-Riemann equations. Then find a function $f(z)$ (with a simple formula in terms of z) so that $f = u + iv$. (And of course, prove/verify that this formula holds.)

Problem 5. II.3, #3. Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$. Suppose that both $f(z)$ and its complex conjugate $\overline{f}(z)$ are analytic on D. Prove that f is constant on D.

Optional Challenges:
A: II.1, #19 (a certain proof of the Fundamental Theorem of Algebra)

B: II.2, #6: Prove that the function $H(z) = \displaystyle\int_0^1 \frac{h(t)}{t - z}\, dt$ (from one of the HW4 challenges) is analytic on the domain $D = \mathbb{C} \setminus [0, 1]$. Compute its derivative.