Math 345, Fall 2024 Professor Rob Benedetto

Homework #5

Due **Monday, September 23** in Gradescope by **11:59 pm ET**

- **WATCH** Video 5: Basic Limit Properties, Part 2
- **WATCH** Video 6: Real Sequences
- **FINISH READING** Section II.1 of Gamelin. Start Section II.2
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. Let $\{a_n\} \subseteq \mathbb{C}$ be a sequences. Prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$

Problem 2. II.1, #1(c): Let $p > 1$. Prove that $\lim_{n\to\infty} \frac{2n^p + 5n + 1}{n^p + 3n + 1} = 2$.

[*Note*: You *may* use theorems from the book or from class.]

Problem 3. II.1, #1(d): Let $z \in \mathbb{C}$. Prove that $\lim_{n\to\infty} \frac{z^n}{n!} = 0$.

Problem 4. II.1, #7: Define a sequence $\{x_0\}_{n\geq 0} \subseteq \mathbb{R}$ inductively by $x_0 = 0$, and $x_{n+1} = x_n^2 + \frac{1}{4}$ for each $n \geq 0$. Prove that $\lim_{n\to\infty} x_n = \frac{1}{2}$.

[*Suggestion*: Prove that the sequence is increasing, and use induction to prove it is bounded. Then it converges (to some real number L) by the Monotone Sequence Theorem, which is the second Theorem on page 35. Now prove that L must be $\frac{1}{2}$.]

Problem 5. Prove that $\mathbb{R}$ is a closed but not open subset of $\mathbb{C}$.

Optional Challenge: II.1, #5: Use Monotone Convergence to prove that the real sequence given by

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

converges to some real number γ satisfying $\frac{1}{2} < \gamma < \frac{3}{5}$.

[*Note*: Gamelin suggests setting $a_n = b_n - \frac{1}{n}$ and proving that $\{b_n\}$ is decreasing, $\{a_n\}$ is increasing, and they have the same limit. This limit γ, known as **Euler's constant** or the **Euler-Mascheroni constant**, arises in the study of the Gamma function $\Gamma(z)$ discussed in Section XIV.1.]