Homework #4

Due **Thursday, September 19** in Gradescope by **11:59 pm ET**

- **WATCH** Video 4: Some Basic Limit Properties (to be posted on moodle soon)
- **READ** Section II.1 of Gamelin, up through p.37
- **READ** the theorem and proof below
- **WRITE AND SUBMIT** solutions to the nine assigned problems in this handout

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Theorem. Let $D \subseteq \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be a function. Then:
f is continuous on D if and only if
for every open set $U \subseteq \mathbb{C}$, the set $f^{-1}(U)$ is also open.

Proof. ($\Longrightarrow$) Given an arbitrary open set $U \subseteq \mathbb{C}$, and given $z_0 \in f^{-1}(U)$, we must find some open disk centered at z_0 that is contained in $f^{-1}(U)$.

Since U is open and $f(z_0) \in U$, there is some $\varepsilon > 0$ such that $D(f(z_0), \varepsilon) \subseteq U$.

Since f is continuous, there is some $\delta_1 > 0$ such that for every $z \in D$ with $0 < |z - z_0| < \delta_1$, we have $|f(z) - f(z_0)| < \varepsilon$.
Since D is open and $z_0 \in f^{-1}(U) \subseteq D$, there is some $\delta_2 > 0$ such that $D(z_0, \delta_2) \subseteq D$.

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. We claim that $D(z_0, \delta) \subseteq f^{-1}(U)$.
To see this, given any $z \in D(z_0, \delta)$, we consider two cases.
If $z = z_0$, then $f(z) = f(z_0) \in U$ and hence $z \in f^{-1}(U)$, as desired.
Otherwise, we have $0 < |z - z_0| < \delta$, and hence $z \in D$ with $|f(z) - f(z_0)| < \varepsilon$.
That is, $f(z) \in D(f(z_0), \varepsilon) \subseteq U$.

We have proven our claim, and therefore $f^{-1}(U)$ is indeed open. QED ($\Longrightarrow$)

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($\Longleftarrow$) Given $z_0 \in D$ and given $\varepsilon > 0$ arbitrary.
Let $U = D(f(z_0), \varepsilon)$, which is an open disk and hence is an open subset of $\mathbb{C}$.
[**NOTE**: this last implication is by Problem 8 on this problem set!]

By hypothesis, then the set $f^{-1}(U) \subseteq D$ is also open.
Moreover, we have $z_0 \in f^{-1}(U)$, since $f(z_0) \in U$.

By definition of open, then, there is some $\delta > 0$ such that $D(z_0, \delta) \subseteq f^{-1}(U)$.

Given any $z \in D$ with $0 < |z - z_0| < \delta$, then, we have $z \in D(z_0, \delta) \subseteq f^{-1}(U)$.
Therefore, $f(z) \in U = D(f(z_0), \varepsilon)$, and hence $|f(z) - f(z_0)| < \varepsilon$. QED

Next, complete the HW problems

found on the next page

Assigned Problems for HW 4

Note: All of the statements I'm asking you to prove below either were Left To Reader in class, or are theorems that Gamelin states but doesn't really prove. So, obviously, you can't just quote unproven results from the book or from class. (But you *may* use statements I gave full proofs of in class, or that were assigned problems on previous homeworks.) Similarly, you can't invoke results from Math 355 or other non-prerequisite courses.

Problems 1–2. Prove that the following functions are continuous on all of $\mathbb{C}$.

1: For any fixed $c \in \mathbb{C}$, the constant function $f(z) = c$.

2: The identity function $g(z) = z$.

Problems 3–6. Prove that the following functions are continuous on all of $\mathbb{C}$.

3: $\operatorname{Re} z$	4: $\operatorname{Im} z$
5: $z \mapsto \lvert z \rvert$	6: $z \mapsto \overline{z}$

[*Suggestion*: use the inequalities listed in the second Example on page 37 as needed.]

Problem 7. Let $D \subseteq \mathbb{C}$, let $f, g : D \to \mathbb{C}$, let $z_0 \in D$, let $L, M \in \mathbb{C}$, and suppose that $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$. Prove that $\lim_{z \to z_0} f(z) \cdot g(z) = LM$.

[*Suggestion*: Refer to the ε-δ proof I did in class about $f(z)/g(z)$ for inspiration.]

Problems 8–9. Let $a \in \mathbb{C}$ and $r > 0$. Prove the following statements:

8: The open disk $D(a, r)$ is indeed an open set.

9: The closed disk $\overline{D}(a, r)$ is indeed a closed set.

Optional Challenges:

A: II.1, #14: If $h : [0, 1] \to \mathbb{C}$ is continuous, where is the function $H(z) = \displaystyle\int_0^1 \frac{h(t)}{t - z}\, dt$ defined? Where is it continuous? (And prove your answers, of course.)

B: Prove that open disks are *not* closed, and that closed disks are *not* open.