Math 345, Fall 2024 Professor Rob Benedetto

Homework #17

Due **Friday, November 22** in Gradescope by **11:59 pm ET**

- **WATCH** Video 20: Isolated Zeros
- **WATCH** Video 21: Laurent Decomposition
- **READ** Sections VI.1 and VI.2 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. VI.1, #1(a). Find all possible Laurent expansions centered at 0 of $\frac{1}{z^2 - z}$

Problem 2. VI.1, #1(c). Find all possible Laurent expansions centered at 0 of $\frac{1}{(z^2 - 1)(z^2 - 4)}$

[*Suggestion*: First write the function as $\frac{A}{z^2 - 1} + \frac{B}{z^2 - 4}$ for some constants A and B.]

Problem 3. VI.2 #1(a). Find all of the isolated singularities (in $\mathbb{C}$, not at ∞) of $f(z) = \frac{z}{(z^2 - 1)^2}$. For each such singularity, determine whether it is removable, essential, or a pole. For each pole, determine its order, and find its principal part.

Problem 4. VI.2 #1(c,e). Find all of the isolated singularities (in $\mathbb{C}$, not at ∞) of the following functions. For each such singularity, determine whether it is removable, essential, or a pole.

(c) $\frac{e^{2z} - 1}{z}$ (e) $z^2 \sin\left(\frac{1}{z}\right)$

Problem 5. VI.2, #7. Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f(z)$, and suppose that there is some $r > 0$ and integer $N \geq 1$ so that $(z - z_0)^N f(z)$ is bounded on $D(z_0, r)$. Prove that z_0 is either removable or else a pole of order at most N.

[*Suggestion*: Riemann's Theorem on Removable Singularities may be useful.]

Optional Challenges:

A. VI.2 #4. Let $s > r > 0$, and suppose f is meromorphic on $D(0, s)$, with only a finite number of poles in this disk, all of which in fact lie in the smaller disk $D(0, r)$. Define $f_1(z)$ to be the sum of the principal parts of $f(z)$ at these poles, and define $f_0 = f - f_1$. Prove that the Laurent decomposition of f on the annulus $\{r < |z| < s\}$ is $f(z) = f_0(z) + f_1(z)$.

B. VI.2, #12. Suppose that z_0 is an isolated singularity of $f(z)$ that is not removable. Prove that z_0 is an essential singularity of $e^{f(z)}$.