Homework #16

Due **Thursday, November 14** in Gradescope by **11:59 pm ET**

- **WATCH** Video 19: Power Series at Infinity
- **READ** Sections V.5 and V.6 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. V.6, #2. Calculate the terms through order five (i.e., up to and including the z^5 term) of the power series expansion centered at $z = 0$ of the function $f(z) = z/\sin z$.
[*Note*: Implicitly, $f(0)$ is the value that makes f continuous (and in fact, analytic) at $z = 0$.]

Problem 2. V.6, #3. Write the power series expansion (centered at 0) of $f(z) = \dfrac{e^z}{1+z}$ as $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$.

(a) Prove that $a_0 = 1$, $a_1 = 0$, and $a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]$ for all $n \geq 2$.

(b) Find the radius of convergence of this series (and of course prove your answer).

Problem 3. V.7, #1(b,c,e). Find the zeros, and the orders of those zeros, of the following functions. (Do not worry about zeros at the point at ∞.)

(b) $\dfrac{1}{z} + \dfrac{1}{z^5}$ **(c)** $z^2 \sin z$ **(e)** $\dfrac{\cos z - 1}{z}$

Problem 4. V.7, #6. Let f be analytic on a domain D, and let $z_0 \in D$. Suppose that $f^{(m)}(z_0) = 0$ for all $m \geq 1$. Prove that f is constant on D.

[**Note**: you cannot simply apply the Theorem on page 144 (about Taylor series) or one of its Corollaries (page 146), since D might not be a disk. You will need to use the results of this section (V.7) somehow.]

Problem 5. V.7, #8. Let f and g be analytic functions on a domain D, and let $z_0 \in D$. Suppose that f has a zero of order $m \geq 0$ at z_0, and g has a zero of order $n \geq 0$ at z_0. Let k be the order of the zero of the function $f(z) + g(z)$ at z_0.

(a) Prove that $k \geq \min\{m, n\}$.

(b) If $m \neq n$, prove that $k = \min\{m, n\}$.

(c) Give an example to show that we *can* have $k > \min\{m, n\}$ in the case that $m = n$.

[**Notes**: First, Gamelin reversed the roles of m and n here, but I decided to keep things in alphabetical order. Second, remember that the statement "f has a zero of order 0 at z_0" means that f does *not* have a zero at z_0, i.e., that $f(z_0) \neq 0$. Third, the constant-zero function has a zero of infinite order at every point; so your proofs should allow for the cases that f is identically zero (i.e., $m = \infty$), or g is identically zero (i.e., $n = \infty$), or both.]

(Optional Challenges On Next Page)

Optional Challenges:

A. V.6, #4. Define the **Bernoulli numbers** B_n by

$$\frac{z}{2}\cot(z/2) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - \cdots = 1 - \sum_{n=1}^{\infty}\frac{B_n}{(2n)!}z^{2n}.$$

(a) Explain why there are no odd-power terms in this series.

(b) Find the radius of convergence of this series

(c) Find the values of B_1, B_2, and B_3.

B. V.7, #9. Suppose f is an analytic function with a zero of order N at z_0, with $1 \leq N < \infty$. Prove that there is some $r > 0$ and an analytic function g on $D(z_0, r)$ so that $f(z) = (g(z))^N$, and such that $g'(z_0) \neq 0$.

C. V.7, #11. Let f be a nonconstant analytic function on a domain D. Prove that f is an **open mapping**, i.e., that for any open set $U \subseteq D$, the image $f(U)$ is also an open set.

[**Hint**: There are two cases. If $f'(z_0) \neq 0$, then use the inverse function theorem, borrowing the fact that the Jacobian of f is $|f'(z_0)|^2$. Otherwise, i.e., if $f'(z_0) = 0$, then use the previous problem.]