Math 345, Fall 2024 Professor Rob Benedetto

Homework #15

Due **Thursday, November 7** in Gradescope by **11:59 pm ET**

- **WATCH** Video 18: Derivatives of Power Series
- **READ** Sections V.4 and V.5 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

---

Problem 1. V.3, #7. Consider the series $\sum_{k=0}^{\infty} \left(2 + (-1)^k\right)^k z^k$.

(a) Use the Cauchy-Hadamard formula to find the radius of convergence of this series.
(b) What happens when the ratio test is applied?
(c) Explicitly evaluate the sum of the series.

Problem 2. V.4 #1(a,b,d). Find the radius of convergence of the power series for each of the following functions, expanding about the indicated point.

(a) $\frac{1}{z-1}$, about $z = i$ (b) $\frac{1}{\cos z}$, about $z = 0$ (d) $\operatorname{Log} z$, about $z = 1 + 2i$

[*Note*: Do **not** actually work out the coefficients of these series. These radii can be found with minimal computation using the results of this section.]

Problem 3. V.4 #2. Prove that the radius of convergence of the power series expansion of $\frac{z^2-1}{z^3-1}$ about $z = 2$ is $R = \sqrt{7}$.

Problem 4. V.4 #3. Find the power series expansion of $\operatorname{Log} z$ about the point $z = i - 2$. Working directly from this series, prove that its radius of convergence is $R = \sqrt{5}$. Explain why this does not contradict the discontinuity of $\operatorname{Log} z$ at $z = -2$.

Problem 5. V.4, #12. Let $f(z)$ be an analytic function with power series expansion $\sum a_n z^n$. If f is an even function (i.e., $f(-z) = f(z)$), prove that $a_n = 0$ for all n odd.
If f is an odd function (i.e., $f(-z) = -f(z)$), prove that $a_n = 0$ for all n even.

Optional Challenges:

A. V.4, #4. Let $f(z)$ be a function that is analytic at $z = 0$ and satisfies $f(z) = z + f(z)^2$. Find the radius of convergence of the power series expansion of f about $z = 0$.

B. V.4, #11. For each integer $n \geq 0$, define $J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{k!(n+k)!2^{n+2k}}$.

Prove that J_n is entire, and that $w = J_n(z)$ satisfies the differential equation

$$w'' + \frac{1}{z}w' + \left(1 - \frac{n^2}{z^2}\right)w = 0$$

[**Note**: J_n is the *Bessel function* of order n, and the differential equation is called *Bessel's differential equation.*]