Math 345, Fall 2024 Professor Rob Benedetto

Homework #14

Due **Monday, November 4** in Gradescope by **11:59 pm ET**

- **WATCH** Video 16: UCC Convergence
- **WATCH** Video 17: Radius of Convergence
- **READ** Sections V.2 and V.3 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. V.2, #7. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of complex numbers. [Recall: this means that there is a real number $M \geq 0$ such that for all $n \geq 1$, we have $|a_n| \leq M$.]

For any $\varepsilon > 0$, prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly on the (closed) half-plane $\operatorname{Re} z \geq 1 + \varepsilon$.

[Here, n^{-z} denotes the principal branch $n^{-z} = e^{-z \operatorname{Log} n}$.]

Problem 2. V.2 #8. Prove that $\displaystyle\sum_{k\geq 1} \frac{z^k}{k^2}$ converges uniformly on the disk $|z| < 1$.

Problem 3. V.3, #1(a,b,d). Find the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{k=0}^{\infty} 2^k z^k$ (b) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{6^k} z^k$ (d) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k z^k}{4^k + 5^k}$

Problem 4. V.3, #5(a). What function is represented by the power series $\displaystyle\sum_{k=1}^{\infty} k z^k$?

Problem 5. V.3, #6. Show that a power series $\sum a_k z^k$, its differentiated series $\sum k a_k z^{k-1}$, and its integrated series $\displaystyle\sum \frac{a_k}{k+1} z^{k+1}$ all have the same radius of convergence.

Optional Challenge: V.3, #4. Show that the function $f(z) = \sum z^{n!}$ is analytic on the open unit disk $|z| < 1$. On the other hand, for every root of unity λ, prove that $\displaystyle\lim_{r\to 1^-} \big|f(r\lambda)\big| = \infty$.

[Thus, $f(z)$ does not extend analytically to any open set larger than the open unit disk. Recall that a root of unity is a complex number $\lambda \in \mathbb{C}$ for which $\lambda^n = 1$ for some integer $n \geq 1$.]