Homework #13

Due **Thursday, October 31** in Gradescope by **11:59 pm ET**

- **WATCH** Video 15: Applying Morera's Theorem
- **READ** Sections IV.5, IV.6, V.1, and V.2 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. IV.5, #2. Let f be an entire function. Suppose that there is a disk $D = D(a, r)$ — that is, the open disk centered at some point $a \in \mathbb{C}$ of some (positive) radius $r > 0$ — such that f does not attain any values in D. (That is, for all $z \in \mathbb{C}$, we have $f(z) \notin D$.) Prove that f is constant.

Problem 2. IV.6, #2. Fix real numbers $a < b$, and let $h : [a, b] \to \mathbb{C}$ be continuous. The *Fourier transform* of h is the function $H : \mathbb{C} \to \mathbb{C}$ given by

$$H(z) = \int_a^b h(t)e^{-itz}\, dt.$$

Prove that H is an entire function, and that there are some positive constants $A, C > 0$ so that

$$\big|H(z)\big| \le Ce^{A|y|} \qquad \text{for all } z = x + iy \in \mathbb{C}.$$

Problem 3. V.1, #5. It is a fact that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to some real number S. Prove that the series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

(which is just a rearrangement of the first series) converges to $3S/2$.

[*Hint*: Organize the terms of the first series in clumps of 4, and organize the terms in the second series in clumps of 3. Then relate each clump in the first series to the corresponding clump in the second series.]

[*Note*: Here are two facts. First, it turns out that $S = \log 2$. Second, Paris is the capital of France. Both of these facts are equally useless for trying to solve this exercise.]

Problem 4. V.2, #10. Let $E_1, \ldots, E_n$ be subsets of $\mathbb{C}$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions that converges uniformly on each of the sets E_j, for $j = 1, \ldots, n$.
Prove that the sequence of functions also converges uniformly on E, where $E = E_1 \cup E_2 \cup \cdots \cup E_n$.

(Optional Challenges On Next Page)

Optional Challenges:

A. IV.5, #3. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *doubly periodic* if there are two different complex numbers ω_0 and ω_1 that do not both lie on the same line through the origin (i.e., such that $\{\omega_0, \omega_1\}$ is linearly independent over $\mathbb{R}$) such that

$$f(z + \omega_0) = f(z + \omega_1) = f(z) \qquad \text{for all } z \in \mathbb{C}.$$

Prove that if f is both entire and doubly periodic, then f is constant.

B. IV.5, #4. Let $n \geq 1$ be an integer and let $R > 0$ be a positive real number. Let f be an entire function such that $f(z)/z^n$ is bounded on the set $|z| \geq R$. Prove that f is a polynomial of degree at most n.

What can be said if $f(z)/z^n$ is bounded on the whole complex plane?