Homework #12

Due **Monday, October 28** in Gradescope by **11:59 pm ET**

- **WATCH** Video 13: Existence of a Primitive
- **WATCH** Video 14: Proof of the Cauchy Differentiation Formula
- **READ** Sections IV.2, IV.3, and IV.4 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. IV.2, #1(a,b), variant. Let γ be *any* piecewise-smooth path in the plane from $-\pi i$ to πi. Use primitives to evaluate the following integrals.

(a) $\displaystyle\int_\gamma z^4\,dz$ (b) $\displaystyle\int_\gamma e^z\,dz$

Problem 2. IV.2, #2. Let γ_1 be *any* piecewise-smooth path in the right half-plane from $-\pi i$ to πi, and let γ_2 be *any* piecewise-smooth path in the left half-plane from $-\pi i$ to πi. For each path γ_j, choose an explicit primitive of $1/z$ (on the right half-plane, and on the left half-plane, respectively). Use this primitive to evaluate $\displaystyle\int_{\gamma_j}\frac{1}{z}\,dz$ for $j = 1, 2$.

Problem 3. IV.3, #4. Use Cauchy's Theorem to prove the key step of the Fundamental Theorem of Algebra — any polynomial with no roots in $\mathbb{C}$ must be constant — using the following strategy. Let $P(z)$ be a polynomial with coefficients in $\mathbb{C}$ that is not constant. (That is, $P(z) \in \mathbb{C}[z]$ with $\deg(P) \geq 1$.) Write $P(z) = P(0) + zQ(z)$ for an appropriate polynomial $Q(z)$, and consider the integral

$$\oint_{|z|=R}\frac{1}{z}\,dz = \oint_{|z|=R}\frac{P(z)}{zP(z)}\,dz = \oint_{|z|=R}\frac{P(0)+zQ(z)}{zP(z)}\,dz = \oint_{|z|=R}\frac{P(0)}{zP(z)}\,dz + \oint_{|z|=R}\frac{Q(z)}{P(z)}\,dz.$$

On the one hand, we can compute the integral on the left side. On the other hand, if P has no roots, you can take the limit as $R \to \infty$ and use the ML-estimate and Cauchy's Theorem to bound the integrals on the right side. So if P has no roots, deduce a contradiction.

Problem 4. IV.4, #1(a,b). Evaluate these integrals using the Cauchy Integral Formula and/or Cauchy Differentiation Formula and/or Cauchy's Theorem.

(a) $\displaystyle\oint_{|z|=2}\frac{z^n}{z-1}\,dz$ for each integer $n \geq 0$ (b) $\displaystyle\oint_{|z|=1}\frac{z^n}{z-2}\,dz$ for each integer $n \geq 0$

Problem 5. IV.4, #1(e,g). Evaluate these integrals using the Cauchy Integral Formula and/or Cauchy Differentiation Formula and/or Cauchy's Theorem.

(e) $\displaystyle\oint_{|z|=1}\frac{e^z}{z^m}\,dz$ for each integer $m \in \mathbb{Z}$ (g) $\displaystyle\oint_{|z|=1}\frac{dz}{z^2(z^2-4)e^z}$

Optional Challenge: IV.3, #6: Let $R > 0$, and suppose $f : \overline{D}(0, R) \to \mathbb{C}$ is continuous on the closed disk $\{|z| \le R\} = \overline{D}(0, R)$ and analytic on the open disk $\{|z| < R\} = D(0, R)$. Prove that $\oint_{|z|=R} f(z)\, dz = 0$.

[*Suggestion*: Approximate f uniformly by $f_r(z) = f(rz)$ for $0 < r < 1$ and let $r \to 1^-$.]