Math 345, Fall 2024 Professor Rob Benedetto

Homework #11

Due **Thursday, October 17** in Gradescope by **11:59 pm ET**

- **WATCH** Video 12: Closed and Exact
- (Optional) **READ** the handout on Green's Theorem on main course website
- **READ** Section IV.1 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. III.2, #5. Fix $0 < a < b$, and let $D = \{z \in \mathbb{C} : a < |z| < b\} \smallsetminus (-b, -a)$, which is a slit annulus. Given points $A, B \in D$ and two paths γ_0, γ_1 from A to B in D, give an explicit homotopy [i.e., what Gamelin calls a family of paths $\gamma_s(t)$] from γ_0 to γ_1 in D.

[*Hint*: Use polar coordinates $re^{i\theta}$, not rectangular coordinates $x + iy$, to describe points in D.]

Problem 2. IV.1, #1(a,b,c). Let γ be the boundary of the triangle $\{0 < x < 1,\ 0 < y < 1-x\}$, oriented counterclockwise. Parametrize γ and use your parametrization to compute each of the following integrals:

$$\text{(a)} \int_\gamma \operatorname{Re}(z)\, dz \qquad \text{(b)} \int_\gamma \operatorname{Im}(z)\, dz \qquad \text{(c)} \int_\gamma z\, dz$$

[*Suggestion*: Once you've computed the integrals in (a) and (b), part (c) is just a linear combination of them.]

Problem 3. IV.1, #3, variant. Fix $R > 0$. Let γ be the circle $\{|z| = R\}$, oriented counterclockwise. Parametrize γ and use your parametrization to compute each of the following integrals for each integer $m = 0, \pm 1, \pm 2, \ldots$.

$$\text{(a)} \int_\gamma z^m\, dz \qquad \text{(b)} \int_\gamma |z|^m\, dz \qquad \text{(c)} \int_\gamma \overline{z}^m\, dz \qquad \text{(d)} \int_\gamma |z^m|\, |dz|$$

[*Suggestion*: Once you've computed the integral in (a), you can use it to do the integrals in (b) and (c) fairly quickly.]

Problem 4. IV.1, #5. Use the ML-estimate to prove that $\left| \oint_{|z-1|=1} \frac{e^z}{z+1}\, dz \right| < 2\pi e^2$.

[*Note*: You need to prove $<$, not $\leq$. Incidentally, in reality, the value of this integral is 0, which is much smaller than $2\pi e^2$. But we won't be able to prove that easily until we learn Cauchy's Theorem in Section IV.3.]

Optional Challenge: IV.1, #9: Let $h(z)$ be a continuous function on a piecewise-smooth curve γ. Prove that the function

$$H(w) = \int_\gamma \frac{h(z)}{z - w}\, dz, \qquad w \in \mathbb{C} \smallsetminus \gamma$$

is analytic on $\mathbb{C} \smallsetminus \gamma$, and find $H'(w)$.