Math 345, Fall 2024 Professor Rob Benedetto

Homework #10

Due **Thursday, October 10** in Gradescope by **11:59 pm ET**

- **WATCH** Video 11: Complex Line Integrals
- **READ** Sections III.1 and III.2 of Gamelin
- **WRITE AND SUBMIT** solutions to the problems in this handout

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Problem 1. III.1, #1(a,c). Compute $\int_\gamma y^2\,dx + x^2\,dy$ along each of the following two paths γ from $(0,0)$ to $(2,4)$:

- γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.
- γ is the vertical line segment $(0,0)$ to $(0,4)$, followed by the horizontal line segment from $(0,4)$ to $(2,4)$.

Problem 2. III.1, #4. Fix $R > 0$, and let γ be the semicircle in the upper half-plane from R to $-R$. Evaluate $\int_\gamma y\,dx$ in two ways: first directly, and then using Green's Theorem.

[*Note*: The path γ is not a closed curve. So to apply Green's Theorem, you'll need to extend it to a closed curve, and separately work out the integral along the extra piece. I'd suggest using the line segment (along the real axis) from $-R$ to R.]

Problem 3. III.1, #5, variant. Let D be a bounded region in the plane with piecewise smooth boundary curve ∂D. Use Green's Theorem to prove that $\int_{\partial D} x\,dy$ is the area of D.

Problem 4. III.2, #1(b,c,d). For each of the following differential forms ω, determine whether ω is independent of path or not.

If yes, find a function h such that $dh = \omega$. If no, find a closed path γ around the origin such that $\int_\gamma \omega \neq 0$.

(b) $\omega = x^2\,dx + y^5\,dy$ (c) $\omega = y\,dx + x\,dy$ (d) $\omega = y\,dx - x\,dy$

Problem 5. III.2, #3. Fix $b > a > 0$, and let D be the annulus $a < |z| < b$. Let $P, Q : D \to \mathbb{R}$ be smooth functions such that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Use Green's Theorem to prove that the value of $\oint_{|z|=r} P\,dx + Q\,dy$ is independent of the radius r, for $a < r < b$.

Optional Challenge:

III.1, #7: Show that both sides of the equation in Green's Theorem are invariant under coordinate change.

[*Note*: See the statement of this problem on page 75 for more details, including the relevant change-of-coordinates formulas both for double integrals and for line integrals.]