Part of the Proof of Green's Theorem

This handout presents a rigorous proof of the "change of variables" step in the proof of Green's Theorem. More precisely, we'll do the following exercise from Gamelin's text (which is the optional challenge problem on Homework 10, in case you want to try it yourself first):

III.1, #7: Do the following step in the proof of Green's Theorem: Show that if the theorem holds for a bounded domain U with piecewise smooth boundary, and if $F : U \to V$ is a smooth bijective map onto another such domain V (so that $F|_{\partial U}$ is also a smooth bijection of ∂U onto ∂V), then the theorem holds for V. Use the change of variables formulae:

$$\int_{\partial V} P\, d\xi = \int_{\partial U} (P \circ F) \cdot \Big(\frac{\partial \xi}{\partial x}\, dx + \frac{\partial \xi}{\partial y}\, dy\Big), \qquad \iint_V R\, d\xi\, d\eta = \iint_U (R \circ F) \cdot \det J_F\, dx\, dy.$$

Note 1: The Greek letters ξ and η are "xi" and "eta", by the way.

Note 2: The multiplication dots $\cdot$ above are both just plain old multiplication, not dot products.

Note 3: We saw the second formula, the change of variables formula for double integrals, on Homework 7, except that there I used the variable names u and v instead of ξ and η, I used the function names F and T instead of P and F, and I used the domain names R and S instead of U and V. Remember that J_F denotes the Jacobian of F.

Note 4: We have *not* previously seen the first formula above, but that's just the corresponding change of variables formula for line integrals, as opposed to double integrals.

Following Gamelin's notation on this problem, we use the coordinates (ξ, η) for points in V.

So $F : U \to V$ inputs $(x, y) \in U$ and outputs $(\xi, \eta) \in V$.

So the function F is given by $F(x, y) = (\xi, \eta)$ where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

Incidentally, note that putting η in place of ξ and P in place of η in the first formula above gives

$$\int_{\partial V} Q\, d\eta = \int_{\partial U} (Q \circ F) \cdot \Big(\frac{\partial \eta}{\partial x}\, dx + \frac{\partial \eta}{\partial y}\, dy\Big),$$

With that preface, let's do the proof:

Proof. Given $P, Q : V \to \mathbb{C}$ [that are C^1 on V and ∂V], let's define $\tilde{P}, \tilde{Q} : U \to \mathbb{C}$ by:

$$\tilde{P}(x, y) = P(F(x, y)) \cdot \frac{\partial \xi}{\partial x}(x, y) + Q(F(x, y)) \cdot \frac{\partial \eta}{\partial x}(x, y), \quad \text{and}$$
$$\tilde{Q}(x, y) = P(F(x, y)) \cdot \frac{\partial \xi}{\partial y}(x, y) + Q(F(x, y)) \cdot \frac{\partial \eta}{\partial y}(x, y),$$

for reasons that will (hopefully) become clear shortly. In particular, by the product rule and the multivariable chain rule — remembering, as noted above, that F inputs the two variables $(x, y) \in U$ and outputs the two variables $(\xi, \eta) \in V$, so that $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ — we have

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial y} &= \Big[\Big(\frac{\partial P}{\partial \xi} \circ F\Big) \cdot \frac{\partial \xi}{\partial y} + \Big(\frac{\partial P}{\partial \eta} \circ F\Big) \cdot \frac{\partial \eta}{\partial y}\Big] \cdot \frac{\partial \xi}{\partial x} + (P \circ F) \cdot \frac{\partial^2 \xi}{\partial y \partial x} \\ &\quad + \Big[\Big(\frac{\partial Q}{\partial \xi} \circ F\Big) \cdot \frac{\partial \xi}{\partial y} + \Big(\frac{\partial Q}{\partial \eta} \circ F\Big) \cdot \frac{\partial \eta}{\partial y}\Big] \cdot \frac{\partial \eta}{\partial x} + (Q \circ F) \cdot \frac{\partial^2 \eta}{\partial y \partial x},\end{aligned}$$

with a similar formula for $\tilde{Q}_x$. In particular, subtracting the two formulae, and cancelling some common terms (including applying Clairaut's theorem to the mixed partials of ξ and η), we have

$$\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y} = \Big(\frac{\partial P}{\partial \eta} \circ F\Big) \cdot \Big(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}\Big) + \Big(\frac{\partial Q}{\partial \xi} \circ F\Big) \cdot \Big(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\Big). \tag{1}$$

Meanwhile, the determinant of the Jacobian of F is

$$\det J_F = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}. \tag{2}$$

Bearing all this in mind, we compute

$$\begin{aligned}
\int_{\partial V} P\,d\xi + Q\,d\eta &= \int_{\partial U} (P \circ F) \cdot \Big(\frac{\partial \xi}{\partial x}\,dx + \frac{\partial \xi}{\partial y}\,dy\Big) + (Q \circ F) \cdot \Big(\frac{\partial \eta}{\partial x}\,dx + \frac{\partial \eta}{\partial y}\,dy\Big) \\
&= \int_{\partial U} \tilde{P}\,dx + \tilde{Q}\,dy = \iint_U \frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y}\,dx\,dy \\
&= \iint_U \Big(\frac{\partial P}{\partial \eta} \circ F\Big) \cdot \Big(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}\Big) + \Big(\frac{\partial Q}{\partial \xi} \circ F\Big) \cdot \Big(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\Big)\,dx\,dy \\
&= \iint_U \Big(\frac{\partial Q}{\partial \xi} \circ F - \frac{\partial P}{\partial \eta} \circ F\Big) \det J_F\,dx\,dy = \iint_V \frac{\partial Q}{\partial \xi} - \frac{\partial P}{\partial \eta}\,d\xi\,d\eta,
\end{aligned}$$

where the first equality is by the change of variables formulae for $\int_{\partial V}$, the second is by definition of $\tilde{P}$ and $\tilde{Q}$, the third is by the truth of Green's Theorem for U, the fourth is by equation (1), the fifth is by the computation of $\det J_F$ in equation (2), and the sixth is by the change of variables formula for $\iint_V$. QED