Proving a Series is NOT Uniformly Convergent

Our principal tools for proving a series converges uniformly are either (a) direct from the definition, or (b) using the Weierstrass M-test.

But what about proving that a convergent series does *not* converge uniformly? Sometimes you might get lucky, and uniform convergence would violate some theorem. (For example, there is a theorem stating that a uniform limit of continuous functions is continuous. So if the sum of a series of continuous functions is *discontinuous*, then the series must not converge uniformly.) But most of the time, you don't get lucky; so you have to return to the (negation of the) definition of uniform convergence.

To illustrate this idea, this handout is devoted to solving the following problem, which I did not assign but which is Exercise #9 in section V.2. In this two-page handout, I've included two incorrect proofs and three correct proofs. Here is the problem:

$$\boxed{\text{Prove that } \sum \frac{z^k}{k} \text{ does not converge uniformly on } D(0,1).}$$

[**Note**: for notation, let's write $f_n(z) = \displaystyle\sum_{k=1}^{n} \frac{z^k}{k}$ and $f(z) = \displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k}$ for the rest of this handout. The comparison and absolute convergence tests can be used to show that $f_n(z) \to f(z)$ for all $z \in D(0,1)$. Thus, *if* $\{f_n\}$ converges uniformly on $D(0,1)$ to *something*, then that something must be $f(z)$.]

Non-proof #1. If $\sum z^k/k$ converged uniformly on $D(0,1)$, then

$$\lim_{z\to 1^-} \sum_{k=1}^{\infty} \frac{z^k}{k} = \sum_{k=1}^{\infty} \Big(\lim_{z\to 1^-} \frac{z^k}{k} \Big) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

which diverges. Contradiction. QED

Commentary: Why is that not a proof? Two reasons. The first (and less important) reason is that the switching of the limit and sum sign has not been justified. If $z = 1$ were in the domain where we were told the sum converges uniformly, the switch would follow from the theorem at the bottom of page 134; but $z = 1$ is not in the domain. Now it turns out to be true that the switch is justifiable even in this case, but that requires proof.

But the second (and bigger) reason that this proof is faulty is that the divergence of the limit does not actually give a contradiction! It is entirely possible to have a sequence of continuous functions h_n that converge uniformly on $D(0,1)$ to h and yet $\lim_{z\to 1^-} h(z)$ diverges. As a really simple example, let $h_n(z) = h(z) = 1/(1-z)$, which is continuous on $D(0,1)$, and of course $h_n \to h$ uniformly (and stupidly) on $D(0,1)$. However, we also have $\lim_{z\to 1^-} h(z) = \infty$.

Both of those issues can be fixed or avoided, but it takes a little more work. Here is a correct proof roughly along those lines.

Proof #1. Note that the power series $f(z) = \sum z^k/k$ is an antiderivative (i.e., primitive) of the power series $g(z) = \sum_{k=0}^{\infty} z^k = 1/(1-z)$ on $D(0,1)$.

Also note that $-\operatorname{Log}(1-z)$ is an antiderivative of $g(z) = 1/(1-z)$ on $D(0,1)$. So we must have $f(z) = C - \operatorname{Log}(1-z)$ for some constant $C \in \mathbb{C}$. In fact, $C = 0$, which can be seen because on the one hand, $f(0) = C - \operatorname{Log}(1) = C$, and on the other hand, $f(0) = 0$ by plugging 0 into the power series. Thus, $f(z) = -\operatorname{Log}(1-z)$.

[Actually, we don't need the precise formula. All we need to know for the rest of the proof is that $\lim_{z\to 1^-} |f(z)| = \infty$, but I think the above paragraph is the easiest way to see that. Besides, I think you ought to know that we're dealing with a completely explicit and familiar function here.]

Assume that $f_n \to f$ uniformly on $D(0,1)$. Then there is some $N \geq 1$ such that for all $n \geq N$ and all $z \in D(0,1)$, we have $|f(z) - f_n(z)| < 1$.
Consider f_N itself, which is a polynomial and hence is continuous on $\overline{D}(0,1)$. Thus, $|f_N(z)|$ is a real-valued continuous function on the (closed and bounded, and therefore compact) set $\overline{D}(0,1)$. It follows that there is some $M \in \mathbb{R}$ such that $|f_N(z)| \leq M$ for all $z \in \overline{D}(0,1)$.
[Note that in the previous two paragraphs we proved facts that hold for *all* $z \in D(0,1)$.]
On the other hand, $\lim_{z\to 1^-} |f(z)| = \lim_{z\to 1^-} |\operatorname{Log}(1-z)| = \infty$.
Thus, there is some $z_0 \in D(0,1)$ such that $|f(z_0)| > 2 + M$.
Combining these three facts and using the triangle inequality, then, we have

$$2 + M < |f(z_0)| \leq |f(z_0) - f_N(z_0)| + |f_N(z_0)| < 1 + M,$$

a contradiction. QED

Non-proof #2. Assume $f_n \to f$ uniformly on $D(0,1)$. So there exists $N \geq 1$ such that for all $n \geq N$ and all $z \in D(0,1)$, we have $|f(z) - f_n(z)| < 1$. Hence, $|f(z)| > |f_n(z)| - 1$ by the triangle inequality.
Then, since f_n is a polynomial, and hence continuous on $\mathbb{C}$, there is some $z_0 \in D(0,1)$ close enough to 1 so that $|f_n(z_0) - f_n(1)| < 1$. Thus, $|f_n(z_0)| > |f_n(1)| - 1$.
Combining the two previous paragraphs, we have $|f(z_0)| > |f_n(1)| - 2$. Taking the limit as $n \to \infty$, we have $|f(z_0)| > \left(\sum \frac{1}{k}\right) - 2 = \infty$, a contradiction. QED

Commentary: Why is that not a proof? This time the problem is better hidden, but equally deadly: note that the choice of z_0 **depends on** n. [If n were made bigger, we'd have to choose z_0 even closer to 1 to force $|f_n(z_0) - f_n(1)| < 1$.] So in the final step, when we take the limit as $n \to \infty$, we would be forced to also let $z_0 \to 1$. Put another way, the argument above falsely assumes that z_0 is fixed forever and doesn't change when we change n; but that's simply not true.
The idea is good, though; here's a corrected proof along the same lines.

Proof #2. Assume that $f_n \to f$ uniformly on $D(0,1)$; thus, there is some $N \geq 1$ such that for all $n \geq N$ and all $z \in D(0,1)$, we have $|f(z) - f_n(z)| < 1$. In particular, for any such n and z, we have

$$|f_n(z) - f_N(z)| \leq |f_n(z) - f(z)| + |f(z) - f_N(z)| < 1 + 1 = 2.$$

Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, its tail $\sum_{k=N+1}^{\infty} \frac{1}{k}$ also diverges. Thus, there is some $n \geq N$ such that $\sum_{k=N+1}^{n} \frac{1}{k} > 4$.
For the sake of notation, set $g(z) = f_n(z) - f_N(z)$, which is a polynomial and hence is continuous. However, by the previous two paragraphs, we have $|g(z)| < 2$ on $D(0,1)$ but $|g(1)| > 4$. Contradiction. [Or, even more formally, by continuity of g at 1, there is some $z \in D(0,1)$ close enough to 1 such that $|g(z) - g(1)| < 1$. Then $4 < |g(1)| \leq |g(1) - g(z)| + |g(z)| < 1 + 2 = 3$, a contradiction.] QED

Here's a third proof, proceeding directly from the definition of (the negation of) uniform convergence, and suggested to me by a former student.

Proof # 3. We will show directly that f_n does not approach f uniformly on $D(0,1)$ by proving that for any $N \geq 1$, there is some $n \geq N$ and some $z \in D(0,1)$ such that $|f(z) - f_n(z)| \geq 1/4$.
Given N, choose $n = N$ and $z = \sqrt[2n]{1/2} \in D(0,1)$. Keeping in mind that this z is a positive real number, we have

$$|f(z) - f_n(z)| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k} \right| = \sum_{k=n+1}^{\infty} \frac{z^k}{k} \geq \sum_{k=n+1}^{2n} \frac{z^k}{k} \geq \sum_{k=n+1}^{2n} \frac{z^{2n}}{2n} = \frac{z^{2n}}{2} = \frac{1}{4},$$

as desired. Here, the second inequality is because all the (positive) terms in that finite sum are at least as big as the last one, $z^{2n}/(2n)$, and the final equality is by our choice of $z = \sqrt[2n]{1/2}$. QED