Solutions to the Final Exam

1. (**20 points**). Let $f(z) = \dfrac{\text{Log}(1-z)}{z^4(z-1)^3}$. Show that f has a pole at $z = 0$. Compute the order of the pole and the principal part of f at $z = 0$.

Solution. Expand f as a Laurent series about $z = 0$ by observing that $-\text{Log}(1-z) = z + z^2/2 + z^3/3 + z^4/4 + \cdots$ (see page 141; it's found by antidifferentiating $1/(1-z)$) for $|z| < 1$.
Meanwhile,

$$\frac{-1}{(z-1)^3} = \frac{1}{2}\frac{2}{(1-z)^3} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) = \frac{1}{2}\frac{d^2}{dz^2}\big(1 + z + z^2 + z^3 + \cdots\big)$$
$$= \frac{1}{2}\big(2 + 6z + 12z^2 + 20z^3 + O(z^4)\big) = 1 + 3z + 6z^2 + 10z^3 + O(z^4)$$

for $|z| < 1$. Thus,

$$f(z) = z^{-4} \cdot \frac{(-1)}{(z-1)^3} \cdot (-\text{Log}(1-z))$$
$$= z^{-4}\big(1 + 3z + 6z^2 + 10z^3 + O(z^4)\big)\left(z + \frac{z^2}{2} + \frac{z^3}{3} + O(z^4)\right)$$
$$= z^{-3} + \left(\frac{1}{2} + 3\right)z^{-2} + \left(\frac{1}{3} + \frac{3}{2} + 6\right)z^{-1} + O(z^0)$$
$$= z^{-3} + \frac{7}{2}z^{-2} + \frac{47}{6}z^{-1} + O(z^0).$$

So f has a $\boxed{\text{pole of order 3}}$ at $z = 0$, with principal part $\boxed{z^{-3} + \frac{7}{2}z^{-2} + \frac{47}{6}z^{-1}}$

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2. (**20 points**). Find all Laurent series centered at 0 for $g(z) = \dfrac{1}{z^3(z^2-6)}$. Specify the domains on which each is equal to g.

Solution. The denominator $z^3(z^2-6)$ is zero at $z = 0, \pm\sqrt{6}$ so that g is analytic on $\mathbb{C} \setminus \{0, \pm\sqrt{6}\}$. Thus, there are two domains on which to consider Laurent decompositions: the punctured open disk $D_1 = \{0 < |z| < \sqrt{6}\}$ and the exterior domain $D_2 = \{|z| > \sqrt{6}\}$.

On D_1, we have $|z^2/6| < 1$, so that $\displaystyle \frac{1}{z^2-6} = -\frac{1}{6}\cdot\frac{1}{1-\frac{z^2}{6}} = -\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{z^2}{6}\right)^k = \sum_{k=0}^{\infty} -\frac{z^{2k}}{6^{k+1}}$

Thus, the Laurent series on D_1 is $\displaystyle g(z) = \frac{1}{z^3}\sum_{k=0}^{\infty} -\frac{z^{2k}}{6^{k+1}} = \boxed{\sum_{k=0}^{\infty} -\frac{z^{2k-3}}{6^{k+1}}} = -\frac{1}{6z^3} - \frac{1}{6^2 z} - \frac{z}{6^3} - \frac{z^3}{6^4} - \cdots$

On D_2, we have $|z^2/6| > 1$, so that $\displaystyle \frac{1}{z^2-6} = \frac{1}{z^2}\cdot\frac{1}{1-\frac{6}{z^2}} = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{6}{z^2}\right)^k = \sum_{k=0}^{\infty}\frac{6^k}{z^{2k+2}}$

Thus, the Laurent series on D_2 is $\displaystyle g(z) = \frac{1}{z^3}\sum_{k=0}^{\infty}\frac{6^k}{z^{2k+2}} = \boxed{\sum_{k=0}^{\infty}\frac{6^k}{z^{2k+5}}} = \frac{1}{z^5} + \frac{1}{6z^7} + \frac{1}{6^2 z^9} + \frac{1}{6^3 z^{11}} + \cdots$

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3. (**25 points**). Given $a \in \mathbb{C}$ with $|a| \neq 1$, compute $\displaystyle\int_{|z|=1} \frac{z+1}{(z-a)^2 \sin z}\, dz$.

Solution. Let $D = D(0,1)$, and let $f(z) = \dfrac{z+1}{(z-a)^2 \sin z}$. Note that f is analytic everywhere on $D \cup \partial D$, except at 0 and a, where it has singularities. (In fact they poles, since f is the reciprocal of an entire function with zeros at a and at every integer multiple of π; but 0 and a are the only such zeros which could be in $D \cup \partial D$.) Since $|a| \neq 1$, there are no singularities on ∂D.
There are three cases: that $a = 0$ (in which case the two poles coincide, possibly changing the residues), that $0 < |a| < 1$ (in which case the poles are distinct but both are inside the contour), and that $|a| > 1$ (in which case the pole at $z = a$ is outside the contour).

Case 1: $a = 0$. We write $z^2 \sin z = z^3 - \frac{1}{3!} z^5 + O(z^7)$, so that

$$f(z) = (1+z)z^{-3} \cdot \frac{1}{1 - z^2/3! + O(z^4)} = \left(z^{-3} + z^{-2}\right)\left(1 + \frac{z^2}{3!} + O(z^4)\right) = z^{-3} + z^{-2} + \frac{1}{6}z^{-1} + \frac{1}{6} + O(z).$$

Thus, f has residue $1/6$ at $z = 0$, and f has no other poles in D. By the residue theorem, then, $\displaystyle\int_{|z|=1} \frac{z+1}{z^2 \sin z}\, dz = \frac{2\pi i}{6} = \boxed{\frac{\pi i}{3}}$

Case 2: $0 < |a| < 1$. Then f has a simple pole at 0 (since $\sin z$ has a simple zero there, and $(z-a)^2$ is not zero there), and f has a double pole at a (since $\sin z$ is nonzero there, and $(z-a)^2$ has a double zero there). We compute

$$\text{Res}[f, 0] = \left.\frac{z+1}{2(z-a)\sin z + (z-a)^2 \cos z}\right|_{z=0} = \frac{0+1}{2(-a)(0) + (-a)^2 \cdot 1} = \frac{1}{a^2}$$

(by Rule 3, page 197), and

$$\text{Res}[f, a] = \left.\frac{d}{dz}\left(\frac{z+1}{\sin z}\right)\right|_{z=a} = \left.\frac{\sin z - (z+1)\cos z}{\sin^2 z}\right|_{z=a} = \frac{\sin a - (a+1)\cos a}{\sin^2 a}$$

by Rule 2. By the residue theorem, then, $\displaystyle\int_{|z|=1} \frac{z+1}{(z-a)^2 \sin z}\, dz = \boxed{2\pi i \left(\frac{1}{a^2} + \frac{1}{\sin a} - \frac{(a+1)\cos a}{\sin^2 a}\right)}$

Case 3: $|a| > 1$. Then the only singularity in D is the simple pole at 0, which, by the calculation of the previous case, has residue $1/a^2$. So $\displaystyle\int_{|z|=1} \frac{z+1}{(z-a)^2 \sin z}\, dz = \boxed{\frac{2\pi i}{a^2}}$

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4. (**15 points**). Let $D \subseteq \mathbb{C}$ be a domain, and let $f, g : D \to \mathbb{C}$ be analytic functions. Suppose that $f(z) \cdot g(z) = 0$ for all $z \in D$. Prove that either $f(z) = 0$ for all $z \in D$ or $g(z) = 0$ for all $z \in D$.

Proof #1. If $f = 0$, then we are done. Otherwise, there is some $z_0 \in D$ such that $f(z_0) \neq 0$. Because f is continuous, there must be some $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subseteq D$, and $f(z) \neq 0$ for all $z \in D(z_0, \varepsilon)$. [FYI: to see that: by continuity of f, there is some $\varepsilon > 0$ such that for all $z \in D$ with $|z - z_0| < \varepsilon$, we have $|f(z) - f(z_0)| < |f(z_0)|/2$. Thus, for all such z, $|f(z)| \leq |f(z_0)| - |f(z_0) - f(z)| < |f(z_0)| - |f(z_0)|/2 = |f(z_0)|/2 > 0$. Hence, $f(z) \neq 0$ for all such z. Now just decrease ε so that $D(z_0, \varepsilon) \subseteq D$.]
However, $f(z)g(z) = 0$ for all $z \in D$, and hence for all $z \in D(z_0, \varepsilon)$. Since $f(z) \neq 0$ for all such z, we have $g(z) = 0$ for all $z \in D(0, \varepsilon)$. Thus, z_0 is a non-isolated zero of g. By the Theorem (about isolated zeros) near the top of page 156, g must be identically zero on D. QED

Proof #2. If $f = 0$, then we are done. Otherwise, there is some $z_0 \in D$ such that $f(z_0) \neq 0$. Pick $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subseteq D$. Write the power series expansions for f and g at z_0 as $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ and $g(z) = \sum_{n\geq 0} b_n(z-z_0)^n$. By the second Corollary on page 146, both series converge on $D(z_0, \varepsilon)$. By equation (6.1) on page 152, $f(z)g(z) = \sum_{n\geq 0} c_n(z-z_0)^n$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$; and this power series also converges on $D(z_0, \varepsilon)$, by the same Corollary. However, $f(z)g(z) = 0$ on D and hence on $D(z_0, \varepsilon)$. By the uniqueness of power series (mentioned in class, but see also the Theorem on page 140), this implies $c_n = 0$ for all $n \geq 0$.

Note that $a_0 = f(z_0) \neq 0$. We will prove by induction that $b_n = 0$ for all $n \geq 0$. Indeed, $a_0 b_0 = c_0 = 0$, so $b_0 = 0$ since $a_0 \neq 0$. Given that $b_0 = \cdots = b_{n-1} = 0$, we have $a_0 b_n = c_n = 0$, which implies $b_n = 0$ since $a_0 \neq 0$, completing the induction. Thus, $g(z) = 0$ for all $z \in D(z_0, \varepsilon)$. Then $g = 0$ on D, by the same Theorem about isolated zeros that we used in Proof #1. QED

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5. (**15 points**). Let $\{a_k\}_{k\geq 0}$ be a sequence in $\mathbb{C} \smallsetminus \{0\}$. The *infinite product* $\prod_{k=0}^{\infty} a_k$ is defined to be

$$\prod_{k=0}^{\infty} a_k = \lim_{N\to\infty} \prod_{k=0}^{N} a_k,$$

if the limit converges **and is nonzero**. (Of course, $\prod_{k=0}^{N} a_k$ means the product $a_0 \cdot a_1 \cdot \cdots \cdot a_N$.)

Prove that if $\prod_{k=0}^{\infty} a_k$ converges (and is nonzero), then $\lim_{k\to\infty} a_k = 1$.

Proof. Let $L = \prod_{k=0}^{\infty} a_k$. By hypothesis, $L \in \mathbb{C} \smallsetminus \{0\}$.

For each $n \geq 0$, let $b_n = \prod_{k=0}^{n} a_k$. Then $\{b_n\}_{n\geq 0}$ is a sequence in $\mathbb{C} \smallsetminus \{0\}$, since we have $a_k \neq 0$ for all k, by hypothesis. In addition, $L = \lim_{n\to\infty} b_n$.

Then we also have $L = \lim_{n\to\infty} b_{n-1}$. Furthermore, for any $n \geq 1$, we have $a_n = b_n / b_{n-1}$. Hence,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{b_k}{b_{k-1}} = \frac{\lim_{k\to\infty} b_k}{\lim_{k\to\infty} b_{k-1}} = \frac{L}{L} = 1.$$

QED

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6. (**30 points**). Use the semicircular contour to compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 \cos x}{x^4 + 4}\, dx$.

Solution. Let $f(z) = \dfrac{z^2 e^{iz}}{z^4 + 4}$, which is analytic except for simple poles at the four fourth roots of -4, i.e., $\pm 1 \pm i$.

We compute $\mathrm{Res}[f, 1+i] = \dfrac{(1+i)^2 e^{i-1}}{4(1+i)^3} = \dfrac{e^{i-1}}{4(1+i)}$ and $\mathrm{Res}[f, -1+i] = \dfrac{(-1+i)^2 e^{-i-1}}{4(-1+i)^3} = \dfrac{e^{-i-1}}{4(-1+i)}$.

Thus, by the Residue Theorem, the integral around the (upper) semicircular contour (of radius $R > \sqrt{2}$) of $f(z)$ is $2\pi i \left(\dfrac{e^{i-1}}{4(1+i)} + \dfrac{e^{-i-1}}{4(-1+i)} \right) = \dfrac{\pi}{2e} \left(\dfrac{e^i + e^{-i}}{2} + \dfrac{e^{-i} - e^i}{2i} \right) = \dfrac{\pi}{2e} \big(\cos(1) - \sin(1) \big)$.

However, for $z = x + iy$ with $y \geq 0$, we have $|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1$. Thus, denoting by γ the semicircular portion of the contour, where $|z| = R$, we have $|f(z)| \leq R^2/(R^4 - 4)$ for z on γ. Thus, by the ML-estimate, $\left|\int_\gamma f(z)\,dx\right| \le \frac{\pi R^3}{R^4-4} = \frac{\pi R^{-1}}{1-4R^{-4}}$, which approaches zero as $R \to \infty$.

Thus, taking the limit as $R \to \infty$, we have $\int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi}{2e}\big(\cos(1) - \sin(1)\big)$. Taking real parts of both sids, we have $\int_{-\infty}^{\infty} \frac{x^2\cos x}{x^4+4}\,dx = \boxed{\frac{\pi}{2e}\big(\cos(1)-\sin(1)\big)}$

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7. (**35 points**). Given real numbers a, b, c with $-1 < a < 1$ and $a \neq 0$, and $b > c > 0$, use the keyhole contour to compute $\int_0^\infty \frac{x^a}{(x+b)(x+c)}\,dx$.

Solution. Let $f(z) = \frac{z^a}{(z+b)(z+c)}$, where we choose a branch of $\log z$ with branch cut along the *positive* real axis $[0,\infty)$ to define z^a; in particular, declare $0 \le \arg z \le 2\pi$. Then $f(z)$ is analytic everywhere except on the branch cut and at the two points $-b, -c$, where f has simple poles. We compute $\operatorname{Res}[f,-b] = \frac{(-b)^a}{c-b}$ and $\operatorname{Res}[f,-c] = \frac{(-c)^a}{b-c}$. Note that $(-b)^a = \exp(a\log b + ia\pi) = e^{ia\pi}b^a$ and similarly $(-c)^a = e^{ia\pi}c^a$. Thus, by the Residue Theorem, the integral around the keyhole contour with outer circle Γ_R and inner circle (traced clockwise) γ_ε, with $0 < \varepsilon < c < b < R$, is $\frac{2\pi i e^{ia\pi}(c^a - b^a)}{b-c}$.

For $|z| = R$, we have $|f(z)| \le \frac{R^a}{(R-b)(R-c)}$. The length of Γ_R is $2\pi R$, and so by the ML-estimate, $\left|\int_{\Gamma_R} f(z)\,dz\right| \le \frac{2\pi R^{a+1}}{(R-b)(R-c)} = \frac{2\pi R^{a-1}}{(1-bR^{-1})(1-cR^{-1})}$, which approaches zero as $R \to \infty$, since $a < 1$.

Similarly, for $|z| = \varepsilon$, we have $|f(z)| \le \frac{\varepsilon^a}{(b-\varepsilon)(c-\varepsilon)}$; and the length of γ_ε is $2\pi\varepsilon$. Thus, by the ML-estimate, $\left|\int_{\gamma_\varepsilon} f(z)\,dz\right| \le \frac{2\pi\varepsilon^{a+1}}{(b-\varepsilon)(c-\varepsilon)}$, which approaches zero as $\varepsilon \to 0$, since $a > -1$.

One of the two horizontal legs of the contour is the integral we want; the other is running backwards (hence $dz = -dx$) and with $\arg z = 2\pi$, so that $z^a = e^{2\pi i a}x^a$. Thus, taking the limit as $R \to \infty$ and $\varepsilon \to 0$ and combining everything above, we get $(1 - e^{2\pi i a})\int_0^\infty f(x)\,dx = \frac{2\pi i e^{ia\pi}(c^a - b^a)}{b-c}$. Thus, the desired integral is $\frac{2\pi i e^{ia\pi}(c^a-b^a)}{(b-c)(1-e^{2\pi i a})} = \frac{2i}{e^{ia\pi} - e^{-ia\pi}} \cdot \frac{\pi(b^a - c^a)}{b-c} = \boxed{\frac{\pi(b^a - c^a)}{(b-c)\sin(\pi a)}}$

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8. (**40 points**). In this problem, you'll compute $\sum_{k=1}^{\infty} \binom{2k}{k+1}\frac{1}{6^k}$. (Recall that $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.)

(a) Show that for every integer $k \ge 1$, $\int_{|z|=1} \frac{(1+z)^{2k}}{z^k}\,dz = 2\pi i\binom{2k}{k+1}$.

(b) Evaluate $\int_{|z|=1} \frac{z\,dz}{z^2 - 4z + 1}$. (*Note*: make sure you know where the poles are!)

(c) Prove that $\sum_{k=0}^{\infty} \frac{(1+z)^{2k}}{(6z)^k}$ converges uniformly to $\frac{-6z}{z^2-4z+1}$ on the circle $|z| = 1$.

(*Hint*: Geometric series. Don't forget to show uniformity!)

(d) Use parts (a)–(c) to evaluate $\sum_{k=1}^{\infty} \binom{2k}{k+1}\frac{1}{6^k}$.

Solution/Proof. (a). The integrand has a pole at $z = 0$ and no other singularities. Expanding $(1+z)^{2k}$, the integrand may be written as:

$$z^{-k}(1+z)^{2k} = z^{-k}\sum_{j=0}^{2k}\binom{2k}{j}z^j = \sum_{j=0}^{2k}\binom{2k}{j}z^{j-k},$$

which has z^{-1} term at $j = k-1$, with coefficient $\binom{2k}{k-1} = \binom{2k}{k+1}$.

By the residue theorem, then, $\displaystyle\int_{|z|=1}\frac{(1+z)^{2k}}{z^k}\,dz = 2\pi i\binom{2k}{k+1}$. QED

(b). Observe that the integrand has poles at the roots of $z^2 - 4z + 1$, which is to say at $z = (4 \pm \sqrt{16-4})/2 = 2 \pm \sqrt{3}$. Now $2+\sqrt{3}$ is outside the contour and $2-\sqrt{3}$ is inside; so by the residue theorem, the integral is $2\pi i$ times the residue at $2-\sqrt{3}$. By Rule 3, we compute

$$\text{Res}\left[\frac{z}{z^2-4z+1}, 2-\sqrt{3}\right] = \frac{z}{2z-4}\bigg|_{z=2-\sqrt{3}} = \frac{2-\sqrt{3}}{-2\sqrt{3}} = \frac{1}{2} - \frac{1}{\sqrt{3}}.$$

So the integral is $\displaystyle\int_{|z|=1}\frac{z\,dz}{z^2-4z+1} = 2\pi i\left[\frac{1}{2} - \frac{1}{\sqrt{3}}\right]$. QED

(c). Set $M_k = (2/3)^k$. By the geometric series test, $\sum M_k$ converges. Moreover, for $|z| = 1$, we observe

$$\left|\frac{(1+z)^{2k}}{(6z)^k}\right| = \frac{|1+z|^{2k}}{6^k} \le \frac{2^{2k}}{6^k} = \left(\frac{4}{6}\right)^k = M_k.$$

Thus, by the M-test, the original sum converges uniformly on the circle.
Moreover, for fixed z on the circle, the series is geometric with ratio $(1+z)^2/(6z)$ and initial term 1. Since $|1+z|^2/|6z| \le 2/3 < 1$, the series converges to

$$\sum_{k=0}^{\infty}\frac{(1+z)^{2k}}{(6z)^k} = \frac{1}{1-\dfrac{(1+z)^2}{6z}} = \frac{6z}{6z-(1+2z+z^2)} = \frac{-6z}{z^2-4z+1}. \qquad \text{QED}$$

(d). Combining the above parts, we compute

$$\begin{aligned}\sum_{k=1}^{\infty}\binom{2k}{k+1}\frac{1}{6^k} &= \sum_{k=1}^{\infty}\frac{1}{2\pi i\cdot 6^k}\int_{|z|=1}\frac{(1+z)^{2k}}{z^k}\,dz = \frac{1}{2\pi i}\int_{|z|=1}\sum_{k=1}^{\infty}\frac{(1+z)^{2k}}{(6z)^k}\,dz\\ &= \frac{1}{2\pi i}\int_{|z|=1} 1 - \frac{6z}{z^2-4z+1}\,dz = \frac{1}{2\pi i}\int_{|z|=1}dz - \frac{6}{2\pi i}\int_{|z|=1}\frac{z}{z^2-4z+1}\,dz\\ &= 0 - \frac{6}{2\pi i}\int_{|z|=1}\frac{z}{z^2-4z+1}\,dz = -\frac{6}{2\pi i}\cdot 2\pi i\left[\frac{1}{2}-\frac{1}{\sqrt{3}}\right] = \frac{6}{\sqrt{3}} - 3 = 2\sqrt{3}-3,\end{aligned}$$

where the first equality is by part (a), the second is by the uniform convergence of part (c) (which, by the Theorem on page 135, allows us to switch the sum and integral signs), the third is by part (c) (observing that we have left out the $k = 0$ term), the fourth is obvious, the fifth is by Cauchy's theorem (since 1 is analytic on the disk), the sixth is by part (b), and the rest is simple algebra. QED

OPTIONAL BONUS. (3 points.) Compute $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+25}$, by **proving** and then applying the following:

Theorem. Let $f(z) = \dfrac{P(z)}{Q(z)}$ be a rational function with $\deg Q \geq 2 + \deg P$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be the distinct zeros of Q. (That is, any of the α_i may be a multiple root of Q, but we have $\alpha_i \neq \alpha_j$ for all $i \neq j$.) Then

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq \text{any } \alpha_j}} f(k) = -\pi \sum_{j=1}^{n} \Big(\text{Res}\left[f(z)\cot(\pi z), \alpha_j \right] \Big).$$

Solution/Proof. First, we prove the Theorem.
Let $g(z) = f(z)\cot(\pi z)$, which has singularities (all poles or removable) at the zeros of Q and at the poles of $\cot(\pi z)$. That is, g has singularities at $\alpha_1, \ldots, \alpha_n$ and at all integers $k \in \mathbb{Z}$.

For each integer $N \geq 1$, let Γ_N be the square with corners $(N + 1/2)(\pm 1 \pm i)$.
Claim 1: For all z on Γ_N, we have $|\cot(\pi z)| \leq 2$.

Proof of Claim 1. Any z on the right edge of Γ_N is of the form $z = (N + 1/2) + iy$ for $y \in [-(N + 1/2), N + 1/2]$. Thus, since $e^{i\pi/2} = i$ and $e^{iN\pi} = \pm 1$, we have
$|\cos(\pi z)| = \dfrac{1}{2}\big|e^{iN\pi}e^{i\pi/2}e^{-\pi y} + e^{-iN\pi}e^{-i\pi/2}e^{\pi y}\big| = \dfrac{1}{2}\big|e^{\pi y} - e^{-\pi y}\big| = \dfrac{1}{2}\big(e^{\pi|y|} - e^{-\pi|y|}\big)$, and
$|\sin(\pi z)| = \dfrac{1}{2}\big|e^{iN\pi}e^{i\pi/2}e^{-\pi y} - e^{-iN\pi}e^{-i\pi/2}e^{\pi y}\big| = \dfrac{1}{2}\big|e^{\pi y} + e^{-\pi y}\big| = \dfrac{1}{2}\big(e^{\pi|y|} + e^{-\pi|y|}\big)$.
Therefore, $|\cot(\pi z)| = \dfrac{e^{\pi|y|} - e^{-\pi|y|}}{e^{\pi|y|} + e^{-\pi|y|}} \leq \dfrac{e^{\pi|y|} + e^{-\pi|y|}}{e^{\pi|y|} + e^{-\pi|y|}} = 1$.
Similarly, we have $|\cot(\pi z)| \leq 1$ for z on the left edge of Γ_N.
Any z on the top edge of Γ_N is of the form $z = x + i(N + 1/2)$ for $x \in [-(N + 1/2), N + 1/2]$. Thus,
$|\cos(\pi z)| = \dfrac{1}{2}\big|e^{(N+1/2)\pi}e^{i\pi x} + e^{-(N+1/2)\pi}e^{-i\pi x}\big| \leq \dfrac{1}{2}\big(e^{(N+1/2)\pi} + e^{-(N+1/2)\pi}\big)$ and
$|\sin(\pi z)| = \dfrac{1}{2}\big|e^{(N+1/2)\pi}e^{i\pi x} - e^{-(N+1/2)\pi}e^{-i\pi x}\big| \geq \dfrac{1}{2}\big(e^{(N+1/2)\pi} - e^{-(N+1/2)\pi}\big)$.
Therefore, $|\cot(\pi z)| \leq \dfrac{1 + e^{-(2N+1)\pi}}{1 - e^{-(2N+1)\pi}} \leq \dfrac{e^{\pi} + 1}{e^{\pi} - 1} \leq 2$.
Similarly, $|\cot(\pi z)| \leq 2$ for z on the bottom edge of Γ_N, and hence on all of Γ_N. QED Claim 1.

Claim 2: $\displaystyle\lim_{N\to\infty} \int_{\Gamma_N} g(z)\,dz = 0$.

Proof of Claim 2. The length of Γ_N is $4(2N + 1)$, and every $z \in \Gamma_N$ has $|z| \geq N + 1/2$.
Meanwhile, since $\deg Q \geq 2 + \deg P$, we have $\displaystyle\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{zP(z)}{Q(z)} = 0$.

Thus, for any $\varepsilon > 0$, there is some $M \geq 0$ so that for all $|z| \geq M$, we have $|f(z) - 0| < \dfrac{\varepsilon}{24}$. Given any integer $N \geq M$, then, and any $z \in \Gamma_N$, we have $|z| > N \geq M$, and hence $|zf(z)| < \varepsilon/27$, and therefore $|zg(z)| \leq \varepsilon/9$. Hence, $|g(z)| \leq \dfrac{\varepsilon}{9|z|} \leq \dfrac{\varepsilon}{9(N + 1/2)}$. Thus, by the ML-estimate,
$\left|\displaystyle\int_{\Gamma_N} g(z)\,dz\right| \leq \dfrac{\varepsilon}{9(N + 1/2)} \cdot \big(4(2N + 1)\big) = \dfrac{8\varepsilon}{9} < \varepsilon.$ QED Claim 2.

For any N large enough that $|\alpha_j| < N + 1/2$ for all $j = 1, \ldots, n$, the Residue Theorem says that

$$\frac{1}{2\pi i} \int_{\Gamma_N} g(z)\,dz = \sum_{\substack{k=-N \\ k\neq \text{any } \alpha_j}}^{N} \text{Res}[g; k] + \sum_{j=1}^{n} \text{Res}[g; \alpha_j].$$

Taking the limit as $N \to \infty$ and invoking Claim 2, we have

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq\text{any }\alpha_j}} \operatorname{Res}[g;k] = -\sum_{j=1}^{n} \operatorname{Res}[g;\alpha_j]. \qquad (\bullet)$$

For any integer k that is not a root α_j of Q, note that $\sin(\pi z)$ has a simple zero at k, and $f(z)\cos(\pi z)$ is analytic at k. Thus, by Rule 3 and the fact that $\frac{d}{dz}\big(\sin(\pi z)\big) = \pi\cos(\pi z)$, we have

$\operatorname{Res}[g;k] = \frac{f(z)\cos(\pi z)}{\pi\cos(\pi z)}\Big|_{z=k} = \frac{f(k)}{\pi}$.

Multiplying equation $(\bullet)$ by π, then, the desired formula is immediate. QED Theorem

Finally, we apply the Theorem. Let $P(z) = 1$ and $Q(z) = z^2 + 9$, so that $f(z) = P(z)/Q(z)$ fits the hypotheses. The two zeros of Q are $\pm 3i$, and each is a simple pole of $g(z) = f(z)\cot(\pi z)$.

By Rule 3 applied to $g(z) = \cot(\pi z)/(z^2 + 25)$, we have

$\operatorname{Res}[g, 5i] = \frac{\cot(\pi z)}{2z}\Big|_{z=5i} = \frac{1}{10i}\cot(5\pi i) = \frac{1}{10i}(-i\coth(5\pi)) = -\frac{1}{10}\coth(5\pi)$,

since $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$, so that $\cot(ix) = -i\coth(x)$.

Similarly, $\operatorname{Res}[g, -5i] = \frac{\cot(\pi z)}{2z}\Big|_{z=-5i} = -\frac{1}{10i}\cot(-5\pi i) = -\frac{1}{10i}(i\coth(5\pi)) = -\frac{1}{10}\coth(5\pi)$.

Summing and multiplying by $-\pi$, the Theorem yields $\sum_{k\in\mathbb{Z}} \frac{1}{k^2+25} = \frac{\pi}{5}\coth(5\pi)$.

That is, $\frac{1}{25} + 2\sum_{n=1}^{\infty}\frac{1}{n^2+25} = \frac{\pi}{5}\coth(5\pi)$. Rearranging, $\sum_{n=1}^{\infty}\frac{1}{n^2+25} = \boxed{\frac{\pi}{10}\coth(5\pi) - \frac{1}{50}}$