Math 345, Fall 2024 Professor Rob Benedetto

Solutions to Midterm Exam 2

1. **(12 points)**. State Morera's Theorem.

Answer. Let $D \subseteq \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be continuous. Suppose that for every closed rectangle $R \subseteq D$ with sides parallel to the axis, we have $\int_{\partial R} f(z)\, dz = 0$. Then f is analytic on D.

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2. **(15 points)**. Compute the integral $\displaystyle\int_{|z|=2024} \frac{\sin z}{z^4}\, dz$.

Solution. Let $f(z) = \sin z$. Then $f'(z) = \cos z$, so $f''(z) = -\sin z$, so $f'''(z) = -\cos z$. Since $z = 0$ lies inside the circle $|z| = 2024$, by CDF, we have

$$\int_{|z|=2024} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!} \cdot f'''(0) = \frac{2\pi i}{6} \cdot (-\cos 0) = \boxed{-\frac{\pi i}{3}}$$

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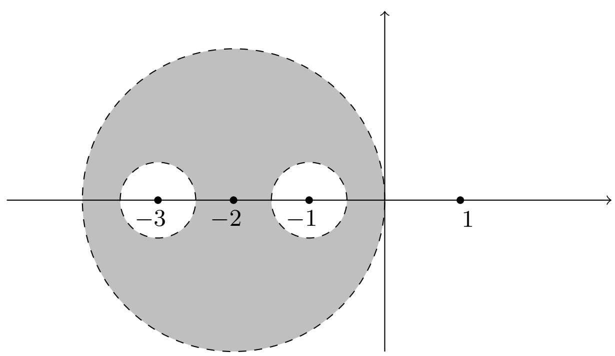
3. **(27 points)**. Compute the integral $\displaystyle\int_{|z+2|=2} \frac{e^{4z}}{(z^2-1)(z+3)}\, dz$.

Solution. Let $f(z)$ be the integrand. Note that two of the singularities of f, at $z = -1$ and $z = -3$, are inside the contour $|z + 2| = 2$, but the third, at $z = 1$, is outside.
Define $D = \{z : |z + 2| < 2 \text{ and } |z + 1| > \frac{1}{2} \text{ and } |z + 3| > \frac{1}{2}\}$, like this:



Define $D_1 = D(-3, \frac{1}{2})$ and $D_2 = D(-3, \frac{1}{2})$. By Cauchy's Theorem applied to D (since f is analytic on D), we have

$$\int_{|z+2|=2} f(z)\, dz = \int_{\partial D_1} f(z)\, dz + \int_{\partial D_2} f(z)\, dz = \int_{|z+3|=\frac{1}{2}} f(z)\, dz + \int_{|z+1|=\frac{1}{2}} f(z)\, dz$$

Write $f(z) = \dfrac{g(z)}{z+3}$ and $f(z) = \dfrac{h(z)}{z+1}$ where $g(z) = \dfrac{e^{4z}}{z^2-1}$ and $h(z) = \dfrac{e^{4z}}{(z-1)(z+3)}$.

By CIF, $\displaystyle\int_{\partial D_1} \frac{g(z)}{z+3} = 2\pi i g(-3) = 2\pi i \frac{e^{-12}}{9-1} = \frac{\pi i}{4e^{12}}$

and $\displaystyle\int_{\partial D_2} \frac{h(z)}{z+1} = 2\pi i h(-1) = 2\pi i \frac{e^{-4}}{(-2)(2)} = \frac{-\pi i}{2e^4}$

Summing, therefore, the original integral is $\boxed{\dfrac{\pi i}{4e^{12}} - \dfrac{\pi i}{2e^4}}$

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4. **(26 points)** Let $f(z) = \dfrac{e^{(2z^3)}}{1 + \sin(z^2)}$.

4a. Compute the power series of $f(z)$ centered at $z = 0$,

up to and including the z^6 term only.

4b. What is the radius of convergence of the power series in part (a), and why?

Solution. (a): We have $e^{(2z^3)} = 1 + 2z^3 + \frac{1}{2}(2z^3)^2 + O(z^7) = 1 + 2z^3 + 2z^6 + O(z^7)$ and

$$1 + \sin(z^2) = 1 + \left(z^2 - \frac{1}{3!}(z^2)^3 + O(z^7)\right) = 1 + z^2 - \frac{1}{6}z^6 + O(z^7) = 1 - \left(-z^2 + \frac{1}{6}z^6\right) + O(z^7).$$

Thus, $$\frac{1}{1+\sin(z^2)} = 1 + \left(-z^2 + \frac{1}{6}z^6\right) + \left(-z^2 + \frac{1}{6}z^6\right)^2 + \left(-z^2 + \frac{1}{6}z^6\right)^3 + O(z^7)$$
$$= 1 - z^2 + \frac{1}{6}z^6 + z^4 - z^6 + O(z^7) = 1 - z^2 + z^4 - \frac{5}{6}z^6 + O(z^7)$$

So $$f(z) = \left(1 + 2z^3 + 2z^6 + O(z^7)\right)\left(1 - z^2 + z^4 - \frac{5}{6}z^6 + O(z^7)\right)$$
$$= \left(1 - z^2 + z^4 - \frac{5}{6}z^6\right) + \left(2z^3 - 2z^5\right) + \left(2z^6\right) + O(z^7) = \boxed{1 - z^2 + 2z^3 + z^4 - 2z^5 + \frac{7}{6}z^6 + O(z^7)}$$

(b): We have $\sin z \neq -1$ for $|z| < \pi/2$, so $1 + \sin(z^2) \neq 0$ for $|z| < \sqrt{\pi/2}$. Therefore, f is analytic on $D(0, \sqrt{\pi/2})$.

On the other hand, the denominator of $f(z)$ is zero at $z = i\sqrt{\pi/2}$ whereas the numerator is not. Thus, f blows up at $z = i\sqrt{\pi/2}$, and hence f has no analytic extension to any disk $D(0, r)$ for $r > \sqrt{\pi/2}$.

Therefore, by a theorem [one of the corollaries of the Taylor series theorem, on page 146, by the way], the radius of convergence of the power series is $\boxed{\sqrt{\frac{\pi}{2}}}$

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5. **(20 points.)** Let $D = \{z \in \mathbb{C} : |z| \leq 3\}$. (I.e., the closed disk of radius 3 centered at the origin.) Prove that the series $\sum_{k=1}^{\infty} \frac{k + z^k}{(z+8)^k}$ converges uniformly on D.

Solution. For any $z \in D$, we have $|z + 8| \geq 8 - |z| \geq 5$. In addition, for any such z and any $k \geq 1$, we have

$$\left|k + z^k\right| \leq k + |z|^k \leq 3^k + 3^k = 2 \cdot 3^k.$$

Thus, for any such z and k, we have $\left|\frac{k + z^k}{(z+8)^k}\right| \leq 2 \cdot \frac{3^k}{5^k}$.

In addition, $\sum_{k\geq 1} 2 \cdot \frac{3^k}{5^k}$ is a geometric series with ratio $r = 3/5$, which satisfies $|r| < 1$; hence, this last series converges by the Geometric Series Test. Therefore, by the M-test, the original series converges uniformly on D. QED

OPTIONAL BONUS. (2 points.) Let f be an entire function, and suppose there is a constant $M \geq 0$ such that $|f(z)| \leq M|z|^2$ for all $z \in \mathbb{C}$. Prove that there is a complex number $a \in \mathbb{C}$ such that $f(z) = az^2$.

Proof. We first claim that $f'''(z) = 0$ for all $z \in \mathbb{C}$. Given any $z \in \mathbb{C}$, then for any $R > |z|$, the point z lies inside the circle $|w| = R$, and hence the Cauchy Differentiation Formula gives

$$f'''(z) = \frac{3!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^4}\, dw.$$

For any $w \in \mathbb{C}$ with $|w| = R$, we have $|w - z| \geq |w| - |z| = R - |z|$, and hence

$$\left|\frac{f(w)}{(w-z)^4}\right| \leq \frac{MR^2}{(R-|z|)^4}.$$

Since the circle has path length $2\pi R$, the ML-estimate yields

$$0 \leq |f'''(z)| \leq \frac{3}{\pi} \cdot \frac{MR^2}{(R-|z|)^4} \cdot 2\pi R = \frac{6MR^3}{(R-|z|)^4} = \frac{6M}{R(1-|z|/R)^4}.$$

This is true for all $R > |z|$. The limit of the expression on the left is 0 as $R \to \infty$. Thus, $0 \leq |f'''(z)|$ is smaller than every positive real number, and hence $f'''(z) = 0$, proving our claim.
Since $(f'')' = 0$, we have $f''(z) = C_2$ by the uniqueness of primitives up to adding constants. Taking a second antiderivative, we similarly have $f'(z) = C_2 z + C_1$. Antidifferentiating again, we have $f(z) = \frac{C_2}{2} z^2 + C_1 z + C_0$.
Using $z = 0$ in our hypothesis gives $|f(0)| \leq 0$, and hence $C_0 = f(0) = 0$. Thus, $f(z) = az^2 + bz$ for some $a, b \in \mathbb{C}$. Note that $f'(0) = b$.
However, for any $\varepsilon > 0$, the Cauchy differentiation formula gives

$$f'(0) = \frac{1!}{2\pi i} \int_{|w|=\varepsilon} \frac{f(w)}{w^2}\, dw.$$

For any $w \in \mathbb{C}$ with $|w| = \varepsilon$, we have

$$\left|\frac{f(w)}{w^2}\right| \leq \frac{M\varepsilon^2}{\varepsilon^2} = M.$$

Since the circle has path length $2\pi\varepsilon$, the ML-estimate yields

$$0 \leq |f'(0)| \leq M\varepsilon.$$

This is true for all $\varepsilon > 0$. Thus, $0 \leq |f'(0)|$ is smaller than every positive real number, and hence $b = f'(0) = 0$.
That is, $f(z) = az^2$. QED