Solutions to (Take-Home) Midterm Exam 1

1. **(10 points)** Find all complex roots of $z^7 + 8iz = 0$.

Solution. The equation is $z(z^6 + 8i) = 0$, so the roots are 0 and the sixth roots of $-8i$. We have $|-8i| = 8$ and $\text{Arg}(8i) = -\pi/2$. Thus, $8i = 8e^{(2\pi i)n - i\pi/2}$ for any $n \in \mathbb{Z}$. Its sixth roots are therefore $\sqrt{2}e^{i\pi(n/3-1/12)}$ for $n = 0, 1, \ldots, 5$. Thus, the roots of the original equation are:

$$\boxed{z = 0,\ \sqrt{2}e^{-i\pi/12},\ \sqrt{2}e^{i\pi/4},\ \sqrt{2}e^{7i\pi/12},\ \sqrt{2}e^{11i\pi/12},\ \sqrt{2}e^{5i\pi/4},\ \sqrt{2}e^{19i\pi/12}}$$

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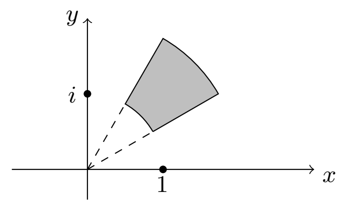
2. **(12 points)** Sketch the region

$$D = \left\{ z \in \mathbb{C} : \frac{\pi}{6} < \text{Arg}\, z < \frac{\pi}{3},\ \text{and } 1 < |z| < 2 \right\}.$$

Then compute and sketch $f(D)$, where $f(z) = iz^3 - 7$.

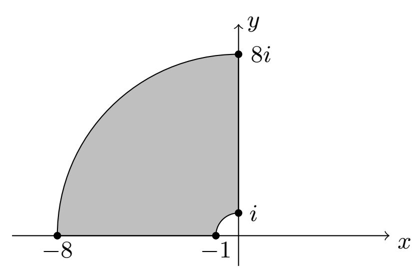
Solution. Here is the region D:



Applying $g(z) = z^3$ cubes the moduli and triples the arguments, so that

$$g(D) = \left\{ z \in \mathbb{C} : \frac{\pi}{2} < \text{Arg}\, z < \pi,\ \text{and } 1 < |z| < 8 \right\},$$

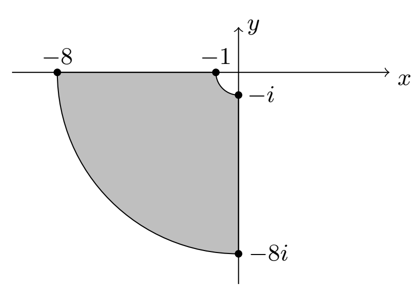
which looks like this:



Multiplying by i rotates 90 degrees counterclockwise, giving

$$ig(D) = \left\{ z \in \mathbb{C} : \pi < \arg z < \frac{3\pi}{2},\ \text{and } 1 < |z| < 8 \right\}.$$

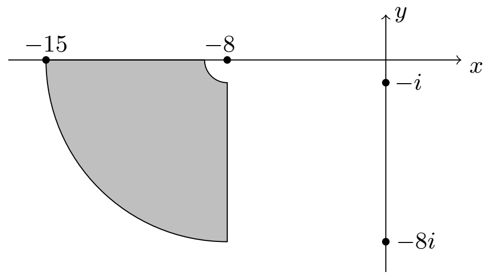
[Note that I had to switch to arg instead of Arg, since the region now crosses the negative real axis.] That looks like this:



Finally, adding -7 translates horizontally left by 7. Thus, we have

$$\boxed{f(D) = \left\{ w \in \mathbb{C} : \pi < \arg(w+7) < \frac{3\pi}{2}, \text{ and } 1 < |w+7| < 8 \right\}}$$

which looks like this (not including the boundary, of course):



Note: The problem did not specifically ask for the two intermediate pictures, but I found it easier to solve the problem by drawing those pictures along the way.

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3. **(14 points)** Let $u(x, y) = x^4 + y^4 + a(x^2y^2 - 2x + 4y) + e^{by}\sin(5x)$.

(a) Find constants $a, b \in \mathbb{R}$ such that u is harmonic on $\mathbb{C}$.

(b) For those choices of a, b, find a harmonic conjugate v for u on $\mathbb{C}$.

(c) For the same choice of a, b, and v, express $f = u + iv$ in terms of $z(= x + iy)$ only.

Solutions. (a): We compute $u_x = 4x^3 + 2axy^2 - 2a + 5e^{by}\cos(5x)$, and so
$u_{xx} = 12x^2 + 2ay^2 - 25e^{by}\sin(bx)$.

Meanwhile, $u_y = 4y^3 + 2ax^2y + 4a + be^{by}\sin(5x)$, and $u_{yy} = 12y^2 + 2ax^2 + b^2e^{by}\sin(5x)$.

Thus, $\Delta u = u_{xx} + u_{yy} = (12 + 2a)(x^2 + y^2) + (b^2 - 25)e^{by}\sin(5x)$.

Choosing $a = -6$ and $b = 5$, then, yields $\Delta u = 0$. [Note: $b = -5$ also works. But I implicitly only asked you to find *one* set of such constants. Your choice. If you did more than one, that's fine, of course.]

In addition, all partial derivatives of u are defined and continuous, since it is made of sums and products of polynomials, exponentials, and trig functions.

Thus, with $\boxed{a = -6 \text{ and } b = 5}$ we have $u = x^4 + y^4 - 6x^2y^2 + 12x - 24y + e^{5y}\sin(5x)$ is harmonic.

(b) Solving $v_y = u_x$ gives $v_y = 4x^3 - 12xy^2 + 12 + 5e^{5y}\cos(5x)$, so
$v = 4x^3y - 4xy^3 + 12y + e^{5y}\cos(5x) + g(x)$, where $g(x)$ is an unknown function.

Thus, $v_x = 12x^2y - 4y^3 - 5e^{5y}\sin(5x) + g'(x)$. Solving $v_x = -u_y$ yields $g'(x) = 24$, so we may choose $g(x) = 24x$. [We may add any constant, but I choose to add 0.]

Hence, $\boxed{v = 4x^3y - 4xy^3 + 12y + 24x + e^{5y}\cos(5x)}$ is a harmonic conjugate for u, since $v_y = u_x$ and $v_x = -u_y$.

(c) We have
$u + iv = x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 + 12x + 12iy + 24ix - 24y + e^{5y}(\sin(5x) + i\cos(5x))$.

That is, $u + iv = (x + iy)^4 + (12 + 24i)(x + iy) + ie^{5y}(\cos(5x) - i\sin(5x))$

$$= z^4 + (12 + 24i)z + ie^{5(y - ix)} = \boxed{z^4 + (12 + 24i)z + ie^{-5iz}}$$

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4. **(14 points)** Let γ_1 be the quarter-circle path (along $|z| = 2$) from 2 to $2i$, and let γ_2 be the straight-line segment from $2i$ to -2. Let $f(z) = 3z - (\overline{z})^2$. Compute

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

Solution. Parametrize γ_1 by $z = 2e^{it}$ for $0 \le t \le \pi/2$, so that $\bar{z} = 2e^{-it}$ and $dz = 2ie^{it}\,dt$. Thus,

$$\int_{\gamma_1} f(z)\,dz = \int_0^{\pi/2} \big[6e^{it} - (2e^{-it})^2\big]2ie^{it}\,dt = 4\int_0^{\pi/2} 3ie^{2it} - 2ie^{-it}\,dt = 4\Big(\frac{3}{2}e^{2it} + 2e^{-it}\Big)\Big|_0^{\pi/2}$$
$$= 4\bigg[\Big(\frac{3}{2}e^{i\pi} + 2e^{-i\pi/2}\Big) - \Big(\frac{3}{2} + 2\Big)\bigg] = 4\Big(-\frac{3}{2} - 2i - \frac{7}{2}\Big) = -20 - 8i$$

Parametrize γ_2 by $z = 2i - (1+i)t$ for $0 \le t \le 2$, so that $\bar{z} = -2i - (1-i)t$ and $dz = -(1+i)\,dt$. Thus,

$$\int_{\gamma_2} f(z)\,dz = \int_0^2 \Big[3\big(2i - (1+i)t\big) - \big(-2i - (1-i)t\big)^2\Big](-1-i)\,dt$$
$$= (-1-i)\int_0^2 \Big[6i - (3+3i)t - \big(-4 + (4+4i)t - 2it^2\big)\Big]\,dt$$
$$= (-1-i)\int_0^2 \big[(4+6i) - (7+7i)t + 2it^2\big]\,dt = (-1-i)\bigg[(4+6i)t - \frac{(7+7i)}{2}t^2 + \frac{2i}{3}t^3\bigg]\bigg|_0^2$$
$$= (-1-i)\bigg[\Big((8+12i) - (14+14i) + \frac{16}{3}i\Big) - (0-0+0)\bigg] = (-1-i)\Big(-6 + \frac{10}{3}i\Big)$$
$$= 6 - \frac{10}{3}i + 6i + \frac{10}{3} = \frac{28}{3} + \frac{8}{3}i.$$

Therefore, the sum of the two integrals is

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = -20 - 8i + \frac{28}{3} + \frac{8}{3}i = \boxed{-\frac{32}{3} - \frac{16}{3}i}$$

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5. **(15 points)** Let $f(z) = \dfrac{z^3 e^{iz}}{z^5 + 10}$, and for any real number $R > 2$, let γ_R be the path from R to $-R$ along the upper half of the circle $|z| = R$. Use the ML-estimate to prove that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0.$$

Proof. Given $R > 2$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \ge 0$ and $|z| = R$, note that $|z^5 + 10| \ge |z^5| - |10| = R^5 - 10$. In addition, writing $z = x + iy$, we have $\left|e^{iz}\right| = \left|e^{ix}e^{-y}\right| = e^{-y} \le 1$, since $y \ge 0$. Furthermore, $|z^3| = R^3$.

Combining these bounds, then, we have

$$|f(z)| = \frac{|z^3| \cdot |e^{iz}|}{|z^5 + 10|} \le \frac{R^3}{R^5 - 10}$$

for all z on the path γ_R.

Meanwhile, the length of the path γ_R is πR. Hence, by the ML-estimate,

$$0 \le \left|\int_{\gamma_R} f(z)\,dz\right| \le \frac{R^3}{R^5 - 10} \cdot \pi R = \frac{\pi R^4}{R^5 - 10} = \frac{\pi R^{-1}}{1 - 10R^{-5}}. \tag{1}$$

The expression on the right side of inequality (1) goes to 0 as $R \to \infty$. Therefore, given any $\varepsilon > 0$, there is some $N > 0$ so that for all $R > N$, we have $\left|\dfrac{\pi R^{-1}}{1 - 10R^{-5}}\right| < \varepsilon$.

Thus, for all $R > N$, we have $\left|\int_{\gamma_R} f(z)\,dz\right| \leq \left|\frac{\pi R^{-1}}{1-10R^{-5}}\right| < \varepsilon$, which proves the desired limit. QED

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6. **(15 points)**

6a. Fix $c \in \mathbb{C}$ with $0 < |c| < 1$. Let $f(z) = \dfrac{z-c}{\bar{c}z-1}$.

Prove that f maps the open unit disk $D(0,1)$ one-to-one and onto itself.

6b. For any $a, b \in D(0,1)$, prove that there is a linear fractional transformation $g(z)$ such that g maps $D(0,1)$ one-to-one and onto itself, with $g(a) = b$.

Proof. (a), Method 1: First we show that f maps $D(0,1)$ *into* itself. Given $z \in D(0,1)$, we have

$$|z-c|^2 = (z-c)(\bar{z}-\bar{c}) = z\bar{z} - c\bar{z} - \bar{c}z + c\bar{c} \qquad \text{and}$$

$$|\bar{c}z-1|^2 = (\bar{c}z-1)(c\bar{z}-1) = c\bar{c}z\bar{z} - c\bar{z} - \bar{c}z + 1.$$

Thus,

$$|\bar{c}z-1|^2 - |z-c|^2 = c\bar{c}z\bar{z} - z\bar{z} - c\bar{c} + 1 = |c|^2|z|^2 - |z|^2 - |c|^2 + 1 = \big(1-|c|^2\big)\big(1-|z|^2\big) > 0,$$

and hence $|z-c| < |\bar{c}z-1|$. That is, $|f(z)| < 1$, so $f(z) \in D(0,1)$, as desired.
Next, since f is an FLT, it has an inverse function, and the FLT inverse formula gives

$$f^{-1}(z) = \frac{-z+c}{-\bar{c}z+1} = \frac{z-c}{\bar{c}z-1} = f(z).$$

Thus, f is one-to-one (since it has an inverse). Moreover, given any $w \in D(0,1)$, we can set $z = f(w)$, and we have $z \in D(0,1)$ by the first thing we proved; so $f(z) = f(f(w)) = w$. Hence, f also maps $D(0,1)$ *onto* itself. QED(a)

(a), Method 2: The points $1, -1, i$ all lie on the circle $|z| = 1$. We have

$$|f(1)| = \left|\frac{1-c}{\bar{c}-1}\right| = \frac{|c-1|}{|\overline{c-1}|} = 1 \qquad \text{and} \qquad |f(-1)| = \left|\frac{-1-c}{-\bar{c}-1}\right| = \frac{|c+1|}{|\overline{c+1}|} = 1,$$

since $|\overline{w}| = |w|$ for all $w \in \mathbb{C}$. In addition, we have

$$\overline{-i(\bar{c}i-1)} = \overline{(\bar{c}+i)} = c - i, \qquad \text{and hence} \qquad \big|\bar{c}i-1\big| = |c-i|,$$

so that

$$|f(i)| = \left|\frac{i-c}{\bar{c}i-1}\right| = \frac{|c-i|}{\big|\bar{c}i-1\big|} = 1.$$

Thus, since all three of $f(1)$, $f(-1)$, $f(i)$ lie on the unit circle $|w| = 1$, and since FLTs map circles and lines to circles and lines, it follows that f maps the unit circle to the unit circle.
In addition, we have $f(0) = \dfrac{-c}{-1} = c$, which lies in the open unit disk, since $|c| < 1$. Thus, since f maps the unit circle to itself and maps (at least one) point inside the disk into the disk, and because f is continuous and bijective on the whole Riemann sphere, it follows that f maps the open unit disk bijectively onto itself. QED (a)

(b): Given $a, b \in D(0,1)$, let $f_a(z) = \dfrac{z-a}{\bar{a}z-1}$, and $f_b(z) = \dfrac{z-b}{\bar{b}z-1}$. Or, if $a = 0$ and/or $b = 0$, define $f_a(z) = z$ and/or $f_b(z) = 0$, respectively. [Alternatively for the $a = 0$ and $b = 0$ cases, part (a) doesn't

actually use the assumption $c \neq 0$, so one can just use f_a and f_b as defined in the first sentence of this part.] Then by part (a), both f_a and f_b map $D(0,1)$ one-to-one and onto itself. Moreover, $f_a(a) = 0$, and $f_b(b) = 0$.
Thus, defining $g = f_b^{-1} \circ f_a$, we see that g is a linear fractional transformation with $g(a) = b$. Moreover, because the composition of one-to-one and onto functions is one-to-one and onto, we also see that g maps $D(0,1)$ one-to-one and onto itself. QED

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7. **(20 points)** Let $f : [3,8] \to \mathbb{C}$ be a continuous function, and let $D = \mathbb{C} \setminus [3,8]$. For all $z \in D$, define

$$g(z) = \int_3^8 \frac{f(t)}{t-z}\,dt.$$

Of course, g is defined on D because for any $z \in D$, the function $f(t)/(t-z)$ is a continuous function of $t \in [3,8]$, and so the integral makes sense.
Prove that g is (complex) differentiable on D. (In fact, g is analytic on D, but I am only asking you to show it's differentiable.)

Proof. By definition, $g'(z) = \lim_{h\to 0} \dfrac{g(z+h)-g(z)}{h} = \lim_{h\to 0} \dfrac{1}{h}\Big(\displaystyle\int_3^8 \frac{f(t)}{t-z-h} - \frac{f(t)}{t-z}\,dt\Big)$

$$= \lim_{h\to 0} \frac{1}{h} \int_3^8 \frac{hf(t)}{(t-z-h)(t-z)}\,dt = \lim_{h\to 0} \int_3^8 \frac{f(t)}{(t-z-h)(t-z)}\,dt.$$

We claim that $g'(z) = \displaystyle\int_3^8 \frac{f(t)}{(t-z)^2}\,dt$. [We guessed this by switching the limit and integral sign.] That is, we claim that

$$\lim_{h\to 0} \int_3^8 \frac{f(t)}{(t-z-h)(t-z)}\,dt = \int_3^8 \frac{f(t)}{(t-z)^2}\,dt.$$

For any fixed $z \in \mathbb{C} \setminus [3,8]$, the integral $\displaystyle\int_3^8 \frac{f(t)}{(t-z)^2}\,dt$ is indeed defined, because the integrand is continuous in the variable $t \in [3,8]$. So it suffices to show the claim.
To prove the claim, note that the absolute value of the difference between the thing we taking the limit of, and our conjectured limit value, is

$$\left| \int_3^8 \frac{f(t)}{(t-z-h)(t-z)} - \frac{f(t)}{(t-z)^2}\,dt \right| = \left| \int_3^8 \frac{hf(t)}{(t-z-h)(t-z)^2}\,dt \right|.$$

We now prove the claim via an epsilon-delta proof, by bounding the above expression.
Given $z \in D$, since D is open, there exists $r > 0$ such that $\overline{D}(z, 2r) \subseteq D$. We can assume in what follows that $|h| < r$, since we are taking $\lim_{h\to 0}$. In particular, $|t-z| \geq 2r$ and

$$|t-z-h| \geq |t-z| - |h| > 2r - r = r \qquad \text{for all } t \in [3,8].$$

The function $|f(t)|$ is a real-valued continuous function on $[3,8]$ (since both f and the absolute value function are continuous). Since $[3,8]$ is closed and bounded, and hence compact, $|f(t)|$ has a maximum value $M \in \mathbb{R}$ on $[3,8]$; clearly $M \geq 0$. Meanwhile, $[3,8]$ has length $L = 5$. By the ML-estimate, then,

$$\left| \int_3^8 \frac{hf(t)}{(t-z-h)(t-z)^2}\,dt \right| \leq \frac{|h| \cdot M}{2r \cdot r^2} \cdot 5 = |h| \cdot \frac{5M}{2r^3}.$$

Given $\varepsilon > 0$, then, let $\delta = \min\left\{ r, \dfrac{2r^3\varepsilon}{5M+1} \right\} > 0$. [Recall we needed $|h| < r$ above; hence the min. And the $+1$ is to avoid dividing by 0.] For any $h \in \mathbb{C}$ with $|h| < \delta$, the above bound shows that

$$\left| \int_3^8 \frac{hf(t)}{(t-z-h)(t-z)^2}\,dt \right| < \frac{2r^3\varepsilon}{5M+1} \cdot \frac{5M}{2r^3} < \varepsilon,$$

proving the claim, and hence finishing the proof. QED

OPTIONAL BONUS. **(2 points)**
Let $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ and } |z - 5i| > 3\}$, i.e., the open upper half-plane with the closed disk $\overline{D}(5i, 3)$ removed.
For any real number $0 < r < 1$, define U_r to be the annulus $U_r = \{z \in \mathbb{C} : r < |z| < 1\}$.
Find a real number $0 < r < 1$ and a function $f : D \to U_r$ that is analytic, one-to-one, and onto.

Answer/Proof. First, let's find an FLT f_1 taking the real line to the unit circle, with the upper half-plane mapping to the interior of the disk. To do this, let's map 0 to 1, 1 to i, and ∞ to -1, which means that traversing $\mathbb{R}$ from left to right corresponds to traversing the circle counterclockwise, so that in both cases the desired region (upper half-plane or disk) is on our left side, giving the desired map. Writing

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1},$$

we have $b_1 = d_1$, $a_1 + b_1 = i(c_1 + d_1)$, and $c_1 = -a_1$. Choosing $d_1 = 1$, this gives

$$f_1(z) = \frac{iz + 1}{-iz + 1}.$$

[As a check, note that $f_1(i) = 0$, so the upper half-plane does indeed map **inside** the unit circle, rather than outside.]

Next, let's figure out what the image $f_1(D)$ is. The image of the upper half-plane, as we said, is the open unit disk $D(0, 1)$. So what is the image of the inner boundary of D, the circle C given by $|z - 5i| = 3$? There are various ways to do this (most obviously, choosing three points on the circle, finding their images, and then figuring out what unique circle or line passes through those three image points), but I'll do the following somewhat sneakier way that involves less computation.

First, I claim that f_1 maps the imaginary axis $\mathbb{R}i$ to the real line $\mathbb{R}$. [In that sentence, and throughout this paragraph, when I say $\mathbb{R}$, I really mean $\mathbb{R} \cup \{\infty\}$.] This is because on the one hand $f_1(-i) = \infty$, which implies that $f_1(\mathbb{R}i)$ is a line (not a circle); on the other hand, since $\mathbb{R}$ and $\mathbb{R}i$ meet at right angles, so must their images. Since $\mathbb{R}$ and $\mathbb{R}i$ meet at 0, and $f_1(0) = 1$, the image $f_1(\mathbb{R}i)$ of the imaginary axis must be the unique straight line passing through 1 and perpendicular to the unit circle. That is, $f_1(\mathbb{R}i) = \mathbb{R}$, as claimed.

Second, the circle C given by $|z - 5i| = 3$ meets $\mathbb{R}i$ at right angles at the two points $2i$ and $8i$. Thus, the image $f_1(C)$ must be the unique circle or line that meets $f_1(\mathbb{R}i) = \mathbb{R}$ at right angles at $f_1(i) = -1/3$ and $f_1(8i) = -7/9$. That is, $f_1(C)$ is the circle C' given by $|z + 5/9| = 2/9$.
[WARNING: note that the **center** of the circle C does **not** map to the center of the circle C'. There was never any claim that FLT's behave that way. They map circles themselves to circles, and they preserve the angles at which curves cross, but they generally do **not** preserve centers of circles.]

Thus, $f_1(D) = D(0, 1) \smallsetminus \overline{D}(-5/9, 2/9)$ is the region formed by removing the smaller (closed) disk from the (open) unit disk. After all, f_1 maps the upper half-plane to $D(0, 1)$, and it maps the *exterior* of $\overline{D}(5i, 3)$ to the exterior of $\overline{D}(-5/9, 2/9)$.

The problem facing us is that the two disks do not share the same center, and hence $f_1(D)$ is not an annulus. So let's try to find another FLT than maps the unit disk to itself BUT moves $\overline{D}(-5/9, 2/9)$ to a disk $\overline{D}(0, r)$ centered at the origin. To do this, let's use

$$f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

and, without yet specifying what $r \in (0,1)$ will be, try to get $f_2(-1) = -1$, $f_2(1) = 1$, $f_2(-7/9) = -r$, and $f_2(-1/3) = r$. That is:

$$\begin{aligned} -a_2 + b_2 &= c_2 - d_2 \\ a_2 + b_2 &= c_2 + d_2 \\ -7a_2 + 9b_2 &= 7rc_2 - 9rd_2 \\ -a_2 + 3b_2 &= -rc_2 + 3rd_2. \end{aligned}$$

Solving the first two equations gives $d_2 = a_2$ and $c_2 = b_2$. Choosing $b_2 = 1$, the last two equations become

$$\begin{aligned} -7a_2 + 9 &= 7r - 9ra_2 \\ -a_2 + 3 &= -r + 3ra_2. \end{aligned}$$

The last equation gives $r + 3 = (3r + 1)a_2$, so that $a_2 = (r + 3)/(3r + 1)$. Substituting this value into the previous equation gives

$$-7(r + 3) + 9(3r + 1) = 7r(3r + 1) - 9r(r + 3), \quad \text{so} \quad 20r - 12 = 12r^2 - 20r.$$

That is, $12r^2 - 40r + 12 = 0$, so that $3r^2 - 10r + 3 = 0$, which factors as $(3r - 1)(r - 3)$.

Thus, r is either 3 or $1/3$. But the choice $r = 3$ is not allowed, since we want $r \in (0,1)$. Therefore, we use $r = 1/3$, giving

$$a_2 = \frac{\frac{1}{3} + 3}{1 + 1} = \frac{5}{3}, \quad \text{and hence} \quad f_2(z) = \frac{\frac{5}{3}z + 1}{z + \frac{5}{3}} = \frac{5z + 3}{3z + 5}.$$

Note that choosing $c = -3/5$ gives $-f_2(z) = (z - c)/(\bar{c}z - 1)$; therefore, by problem 6(a) on this exam, $-f_2$ maps the open unit disk $D(0,1)$ bijectively onto itself. Since $w \mapsto -w$ also maps the open unit disk $D(0,1)$ bijectively onto itself, it follows that f_2 also maps the open unit disk $D(0,1)$ bijectively onto itself.

In addition, since f_2 has real coefficients, it maps real numbers to real numbers, and hence it maps the real line to itself. Since the circle C' (which, recall, is given by $|z + 5/9| = 2/9$) crosses $\mathbb{R}$ at right angles at the points $-7/9$ and $-1/3$, it follows that $f_2(C')$ must be another circle crossing $\mathbb{R}$ at right angles at the points $f(-7/9) = -1/3$ and $f(-1/3) = 1/3$. That is, $f_2(C')$ is the circle C'' centered at 0 of radius $1/3$.

Recall that $f_1(D)$ is the set of points inside the circle C (given by $|z| = 1$) and outside the circle C'. Because $0 \in f_1(D)$, and because $f_2(0) = 3/5$ lies inside the unit disk but outside the circle C'', it follows that $f_2(f_1(D))$ is the annulus inside C and outside C''.

That is, $f_2 \circ f_1$ maps D into the annulus $U_{1/3}$. So the map

$$f(z) = f_2 \circ f_1(z) = \frac{2iz + 8}{-2iz + 8} = \frac{iz + 4}{-iz + 4}$$

maps D bijectively and analytically onto U_r where $r = 1/3$.