

Solutions to Homework #8

Problem 1. II.5, #1(a), variant. Prove that $u(x, y) = x^2 - y^2$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations. (Then $u + iv$ is necessarily analytic, so u is harmonic.)

We have $v_y = u_x = 2x$, so $v = 2xy + h(x)$ for some function $h(x)$.

Then $v_x = 2y + h'(x)$, but since $v_x = -u_y = -(-2y) = 2y$, it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $\boxed{v(x, y) = 2xy}$ is a harmonic conjugate of u .

The associated analytic function f has $f(x + iy) = x^2 - y^2 + 2xyi = (x + iy)^2$, so $\boxed{f(z) = z^2}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 2. II.5, #1(b), variant. Prove that $u(x, y) = xy + 3x^2y - y^3$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

We have $v_y = u_x = y + 6xy$, so $v = \frac{1}{2}y^2 + 3xy^2 + h(x)$ for some function $h(x)$.

Then $v_x = 3y^2 + h'(x)$, but since $v_x = -u_y = -(x + 3x^2 - 3y^2) = -x - 3x^2 + 3y^2$, it follows that $h'(x) = -x - 3x^2$. Thus, we may use $h(x) = -\frac{1}{2}x^2 - x^3$.

So $\boxed{v(x, y) = \frac{1}{2}(y^2 - x^2) + 3xy^2 - x^3}$ is a harmonic conjugate of u .

The associated analytic function f has

$$f(x + iy) = \frac{1}{2}(2xy - i(x^2 - y^2)) + (-y^3 + 3xy^2i + 3x^2y - x^3i) = \frac{-i}{2}(x + iy)^2 - i(x + iy)^3.$$

So $\boxed{f(z) = \frac{-i}{2}z^2 - iz^3}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 3. II.5, #1(d), variant. Prove that $u(x, y) = e^{x^2 - y^2} \cos(2xy)$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

We have $v_y = u_x = 2xe^{x^2 - y^2} \cos(2xy) - 2ye^{x^2 - y^2} \sin(2xy)$,
so $v = e^{x^2 - y^2} \sin(2xy) + h(x)$ for some function $h(x)$.

[**Note:** After taking the derivative u_x , I guessed-and-checked how to do the antiderivative of v_y . You could also use integration by parts, but that would take some time.]

Then $v_x = 2xe^{x^2-y^2} \sin(2xy) + 2ye^{x^2-y^2} \cos(2xy) + h'(x)$,

but since $v_x = -u_y = -(x + 3x^2 - 3y^2) = -(2xe^{x^2-y^2} \sin(2xy) - 2ye^{x^2-y^2} \cos(2xy))$,

it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $v(x, y) = e^{x^2-y^2} \sin(2xy)$ is a harmonic conjugate of u .

The associated analytic function f has

$$f(x + iy) = e^{x^2-y^2} (\cos(2xy) + i \sin(2xy)) = e^{(x^2-y^2)+2xyi} = e^{(z^2)}.$$

So $f(z) = e^{z^2}$ is analytic with $u = \operatorname{Re} f$.

QED

Problem 4. II.5, #1(f), variant. Prove that $u(x, y) = x/(x^2 + y^2)$ is harmonic on $\mathbb{C} \setminus \{0\}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous on $\mathbb{C} \setminus \{0\}$. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

$$\text{We have } u_x = \frac{1(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Seeing how that derivative worked out, it's easy to guess-and-check that with $v_y = u_x$, we have

$$v = \frac{-y}{x^2 + y^2} + h(x) \text{ for some function } h(x).$$

$$\text{Then } v_x = \frac{2xy}{(x^2 + y^2)^2} + h'(x),$$

but since $v_x = -u_y = \frac{2xy}{(x^2 + y^2)^2}$, it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $v(x, y) = \frac{-y}{x^2 + y^2}$ is a harmonic conjugate of u .

The associated analytic function f has

$$f(x + iy) = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

So $f(z) = \frac{1}{z}$ is analytic with $u = \operatorname{Re} f$.

QED

Problem 5. II.6, #4. Let $A > 0$ be a positive real number. Find a conformal map of the horizontal strip $\{-A < \operatorname{Im} z < A\}$ onto the right half-plane $\{\operatorname{Re} w > 0\}$. (And of course, verify/prove all your claims.)

Solution. Recall that e^z maps the horizontal strip $\{-\pi/2 < \operatorname{Im} z < \pi/2\}$ to the right half-plane, since $\operatorname{Arg}(e^z) = \operatorname{Im} z$, and $|e^z| = e^{\operatorname{Re} z}$.

And of course, multiplication by $\pi/(2A)$ maps the strip $\{-A < \operatorname{Im} z < A\}$ onto the strip $\{-\pi/2 < \operatorname{Im} z < \pi/2\}$.

Composing these two maps, then, the analytic function $f(z) = e^{(\pi/(2A))z}$ maps the strip $\{-A < \operatorname{Im} z < A\}$ to the half-plane $\{\operatorname{Re} w > 0\}$.

In addition, $f'(z) = \frac{\pi}{2A} e^{(\pi/(2A))z}$ is never 0. Therefore, by the Theorem on page 59, f provides the desired conformal map.

[Note: There are other correct answers, but this is probably the easiest one to write down.]