

Solutions to Homework #7

Problem 1. II.4, #2. Let $a \in \mathbb{C} \setminus \{0\}$ be a nonzero complex number, and Let $f(z)$ be an analytic branch of z^a . Prove that $f'(z) = af(z)/z$.

Proof. We have $f(z) = \exp(ag(z))$ where $\exp(z) = e^z$, and $g(z)$ is some (analytic) branch of $\log z$.

Note that the domain D of g is also the domain of f . For any $z \in D$, we have $g'(z) = \frac{1}{z}$. (See the sentence just after equation (4.2) on page 52.)

Therefore, since $\frac{d}{dz}(\exp(z)) = \exp(z)$, at any $z \in D$, the Chain Rule gives us

$$f'(z) = \exp(ag(z)) \cdot ag'(z) = f(z) \cdot a \cdot \frac{1}{z} = \frac{af(z)}{z} \quad \text{QED}$$

Problem 2. II.4, #7. Let $D \subseteq \mathbb{C}$ be a bounded domain, and let f be a bounded analytic function on D . Suppose also that f is one-to-one on D . Prove that $\text{Area}(f(D)) = \iint_D |f'(z)|^2 dx dy$.

Proof. From the Theorem on page 51, the Jacobian of f , viewed as a map from D to $\mathbb{R}^2$, satisfies $|\det J_f| = |f'(z)|^2$. Therefore, by the Multivariable Change-of-Variables Formula,

$$\text{Area}(f(D)) = \iint_{f(D)} 1 du dv = \iint_D 1 \cdot |f'(z)|^2 dx dy = \iint_D |f'(z)|^2 dx dy. \quad \text{QED}$$

Problem 3. II.4, #9. Let $D = D(0, 1)$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and let $f(z) = z^2$. Compute $\iint_D |f'(z)|^2 dx dy$. Interpret the answer in terms of areas; that is, explain how the value you get actually agrees with the previous problem.

Solution. We have $f'(z) = 2z$, so $|f'(z)|^2 = 4|z|^2 = 4(x^2 + y^2)$. We now have a Math 211-style double integral, which we compute with polar coordinates:

$$\begin{aligned} \iint_D |f'(z)|^2 dx dy &= \iint_D 4(x^2 + y^2) dx dy = \int_0^{2\pi} \int_0^1 4r^2 \cdot r dr d\theta = \int_0^{2\pi} \int_0^1 4r^3 dr d\theta \\ &= \int_0^{2\pi} r^4 \Big|_0^1 d\theta = \int_0^{2\pi} (1 - 0) d\theta = \theta \Big|_0^{2\pi} = \boxed{2\pi} \end{aligned}$$

Interpretation. The image disk $f(D)$ is the same disk $D(0, 1)$, which has area π , not 2π . But of course, the function $f : D \rightarrow f(D)$ is not one-to-one, but rather two-to-one, so the previous problem does not apply directly. Instead, since the mapping is two-to-one, we get 2 times the area as the value of the integral.

More precisely, let's cut D into two pieces, the right half $U_1 = \{z \in D : \text{Re } z > 0\}$ and the left half $U_2 = \{z \in D : \text{Re } z < 0\}$. Then each half maps one-to-one onto $D(0, 1)$ (well, aside from removing a slit of area 0). So the contribution to the integral above from each of U_1 and U_2 alone is π , for the area π of its image $D(0, 1)$ (minus the slit), totaling 2π .

Problem 4. II.5, #2. Let $D \subseteq \mathbb{C}$ be a domain, and let $u, v : D \rightarrow \mathbb{R}$ be harmonic functions. Suppose that v is a harmonic conjugate for u . Prove that $-u$ is a harmonic conjugate for v .

Proof. Let $f = u + iv$, which is analytic on D because v is a harmonic conjugate for u . Let $g = -if = v - iu$. Then (as a constant multiple of f), g is also analytic. Since $v = \text{Re } g$, then by definition of harmonic conjugate, we have that $-u = \text{Im } g$ is a harmonic conjugate for v . QED

Problem 5. II.5, #3(a,b), slight variant. Define $u(z) = \begin{cases} \operatorname{Im}(1/z^2) & \text{for } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{for } z = 0. \end{cases}$

Prove that all four of $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$ exist at *all* points of $\mathbb{C}$ (viewed as $\mathbb{R}^2$), including at the point $(0,0)$.

Then verify that u satisfies Laplace's equation on $\mathbb{R}^2$. That is, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ everywhere on $\mathbb{R}^2$.

Proof. Since $u(z) = \operatorname{Im}(\bar{z}^2/|z|^4)$ for $z \neq 0$, a quick computation shows that $u(x,y) = \frac{-2xy}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$.

Away from $(0,0)$, $u(x,y)$ is a rational function defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$, so (by the quotient rule, etc.), its partial derivatives u_x and u_y are also rational functions defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, *their* partial derivatives u_{xx} and u_{yy} are also defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$. [Note: you can also compute their formulas exactly, but we won't need to here.]

Note that $u(x,0) = 0$ for all $x \in \mathbb{R}$, and $u(0,y) = 0$ for all $y \in \mathbb{R}$. Thus, by definition of partial derivatives, for every $x, y \in \mathbb{R}$, we have

$$u_x(x,0) = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x,0) - u(x,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0$$

and

$$u_y(0,y) = \lim_{\Delta y \rightarrow 0} \frac{u(0,y+\Delta y) - u(0,y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0.$$

In particular, both $u_x(0,0)$ and $u_y(0,0)$ exist and equal 0.

Since $u_x(x,0)$ is identically zero on the x -axis, it follows that $u_{xx}(x,0)$ is also identically zero. In particular, $u_{xx}(0,0)$ exists and equals 0. Similarly, $u_{yy}(0,y)$ is identically zero, and in particular, $u_{yy}(0,0)$ exists and equals 0. We have shown that u_x , u_y , u_{xx} , and u_{yy} are defined at *all* points of $\mathbb{R}^2$, as desired.

For Laplace's equation, note that at $(0,0)$, we have just shown that $u_{xx}(0,0) = u_{yy}(0,0) = 0$, and hence $u_{xx}(0,0) + u_{yy}(0,0) = 0$. Away from the origin, $u = \operatorname{Im}(1/z^2)$ is the imaginary part of an analytic function (namely $f(z) = 1/z^2$) and hence is harmonic on $\mathbb{C} \setminus \{0\}$ as well. [This can also be confirmed by computing u_{xx} and u_{yy} directly, but that's much more painful.] Thus, u satisfies Laplace's equation on all of $\mathbb{R}^2$. QED

Side note: If you really want to compute derivatives, then away from $(0,0)$, direct computation gives

$$u_x(x,y) = \frac{-2y(x^2+y^2)^2 + 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{2y(-x^2-y^2+4x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\text{and similarly } u_y = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3}.$$

Another pair of brute-force computations then gives $u_{xx} = \frac{24xy(y^2-x^2)}{(x^2+y^2)^4}$ and $u_{yy} = \frac{24xy(x^2-y^2)}{(x^2+y^2)^4}$ away from $(0,0)$.

And those two expressions for u_{xx} and u_{yy} clearly sum to 0 for $(x,y) \neq (0,0)$. But even if you do all of these explicit computations, you still need to go back and compute all of these partial derivatives at $(0,0)$ by the limit definition, as in the main proof above.

Problem 6. II.5, #3(c,d), slight variant. With u as in the previous problem, prove that $\frac{\partial^2 u}{\partial x \partial y}$ does *not* exist at $(0,0)$. Conclude that u is *not* harmonic on the whole plane, even though we just saw that it satisfies Laplace's equation on the whole plane.

Proof. This time, we *do* need to compute at least u_y explicitly away from $(0,0)$, giving:

$$u_y(x, y) = \frac{-2x(x^2 + y^2)^2 + 2xy \cdot 2(x^2 + y^2)(2y)}{(x^2 + y^2)^4} = \frac{2y(-x^2 - y^2 + 4y^2)}{(x^2 + y^2)^3} = \frac{2y(3y^2 - x^2)}{(x^2 + y^2)^3}$$

As we saw in the previous problem, we also have $u_y(0,0) = 0$. Thus,

$$u_y(x, y) = \begin{cases} \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

Restricting to $y = 0$, we have

$$u_y(x, 0) = \begin{cases} -2x^{-3} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is not continuous and hence not differentiable at $x = 0$, since $\lim_{x \rightarrow 0} \frac{-2}{x^3}$ (for $x \in \mathbb{R}$) diverges. Thus, $u_{yx}(0,0)$ does not exist, as desired.

In addition, it follows that u does *not* have all of its second partials existing, so u is *not* harmonic on the whole plane. QED

Side note: Actually, the original function u isn't even *continuous* on the plane.

Indeed, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u(x, x) = \frac{-2x^2}{(x^2 + x^2)^2} = -\frac{1}{2x^2}$.

Thus, $\lim_{x \rightarrow 0} u(x, x) = \lim_{x \rightarrow 0} -\frac{1}{2x^2} = -\infty$ diverges, so indeed, u is not even continuous at $(0,0)$.