

Solutions to Homework #6

Problem 1. II.1, #16, slight variant: Prove that

- (a) the slit plane $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped, but
- (b) the punctured plane $\mathbb{C} \setminus \{0\}$ is *not* star-shaped.

Proof. (a) Let $D = \mathbb{C} \setminus (-\infty, 0]$. We will show that $z_0 = 1 \in D$ works as the star point of this domain.

Given any $z \in D$, write $z = x + iy$. To prove that the line segment L from 1 to z is contained in D , we consider two cases.

Case 1. Suppose $y = 0$. Then $z \in \mathbb{R}$, and hence we must have $z > 0$. Thus, the line segment L is $[1, z]$ if $z \geq 1$, or $[z, 1]$ if $0 < z < 1$. Either way, L is contained in the positive real line $(0, \infty)$, and hence it is contained in D .

Case 2. Otherwise, we have $y \neq 0$. Then every point w on L besides the endpoint $z_0 = 1$ has nonzero imaginary part, and hence $w \in D$. Since $1 \in D$ as well, it follows that $L \subseteq D$. QED (a)

(b): Let $D' = \mathbb{C} \setminus \{0\}$. Given any $z_0 \in D'$, we must show that z_0 does not serve as a star point of D' .

Let $z = -z_0$. Then $z \neq 0$, since $z_0 \neq 0$; that is, $z \in D'$. However, the line segment L from z_0 to z contains the point $0 \notin D'$. Thus, $L \not\subseteq D'$, proving that z_0 is not a star point of D' . QED (b)

Problem 2. II.2, #2. For any $z \in \mathbb{C} \setminus \{1\}$ and any integer $n \geq 1$, prove that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - z^n}{(1 - z)^2} - \frac{nz^n}{1 - z}.$$

Proof. (Method 1): By induction on $n \geq 1$.

For $n = 1$, the left side is 1, and the right side is $\frac{1 - z}{(1 - z)^2} - \frac{z}{1 - z} = \frac{1}{1 - z} - \frac{z}{1 - z} = \frac{1 - z}{1 - z} = 1$, proving the base case.

Assuming the equality holds for some particular $n \geq 1$, we must show it also holds for $n + 1$. We have

$$\begin{aligned} 1 + 2z + 3z^2 + \cdots + nz^{n-1} + (n+1)z^n &= \left(\frac{1 - z^n}{(1 - z)^2} - \frac{nz^n}{1 - z} \right) + (n+1)z^n \quad [\text{by the inductive assumption}] \\ &= \frac{1 - z^n}{(1 - z)^2} + \left(-\frac{nz^n}{1 - z} + (n+1)z^n \right) = \frac{1 - z^{n+1} + z^{n+1} - z^n}{(1 - z)^2} + \frac{(-nz^n) + (n+1)z^n - (n+1)z^{n+1}}{1 - z} \\ &= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{z^n(1 - z)}{(1 - z)^2} + \frac{z^n - (n+1)z^{n+1}}{1 - z} = \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{z^n}{1 - z} + \frac{z^n}{1 - z} - \frac{(n+1)z^{n+1}}{1 - z} \\ &= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{(n+1)z^{n+1}}{1 - z} \end{aligned} \quad \text{QED}$$

(Method 2): For any such z and n , we have

$$(1 - z)(1 + z + z^2 + \cdots + z^{n-1}) = (1 + z + z^2 + \cdots + z^{n-1}) - (z + z^2 + z^3 + \cdots + z^n) = 1 - z^n.$$

Therefore, we have

$$(1 - z)^2(1 + 2z + 3z^2 + \cdots + nz^{n-1}) = (1 - z) \left[(1 + 2z + 3z^2 + \cdots + nz^{n-1}) - (z + 2z^2 + 3z^3 + \cdots + nz^n) \right]$$

$$= (1 - z) \left[(1 + z + z^2 + \cdots + z^{n-1}) - nz^n \right] = (1 - z^n) - nz^n(1 - z).$$

Dividing both sides by $(1 - z)^2$, then, we have $1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - z^n}{(1 - z)^2} - \frac{nz^n}{1 - z}$. QED

(Method 3): Start the same way as Method 2, to get:

$$(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}, \text{ and hence } 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Both sides are analytic on $\mathbb{C} \setminus \{1\}$, so we may differentiate, to get:

$$\begin{aligned} 1 + 2z + 3z^2 + \cdots + nz^{n-1} &= \frac{-(n+1)z^n(1-z) - (1-z^{n+1})(-1)}{(1-z)^2} \\ &= \frac{-nz^n(1-z) - z^n + z^{n+1} + 1 - z^{n+1}}{(1-z)^2} = \frac{-nz^n(1-z) + 1 - z^n}{(1-z)^2} = \frac{1 - z^n}{(1-z)^2} - \frac{nz^n}{1-z} \end{aligned} \quad \text{QED}$$

Problem 3. II.2, #3. Prove (from the definition) that the functions $\text{Re}(z)$ and $\text{Im}(z)$ are not differentiable at any point.

Proof. For any $z \in \mathbb{C}$, the limit definition of the derivative of $\text{Re}(z)$ is

$$\text{Re}'(z) = \lim_{h \rightarrow 0} \frac{\text{Re}(z+h) - \text{Re}(z)}{h} = \lim_{h \rightarrow 0} \frac{\text{Re } h}{h}.$$

Writing $h = u + iv$, first let $v = 0$, so that $h = u$. Then the limit above (along the horizontal line $h \in \mathbb{R}$) is

$$\lim_{u \rightarrow 0} \frac{\text{Re}(u + 0i)}{u + 0i} = \lim_{u \rightarrow 0} \frac{u}{u} = \lim_{u \rightarrow 0} 1 = 1$$

But if we use $u = 0$, so that $h = iv$, then the limit above (along the vertical line $h \in i\mathbb{R}$) is

$$\lim_{v \rightarrow 0} \frac{\text{Re}(0 + iv)}{0 + iv} = \lim_{v \rightarrow 0} \frac{0}{iv} = \lim_{v \rightarrow 0} 0 = 0.$$

Since these two limits disagree, the overall limit defining $\text{Re}'(z)$ diverges. QED Re

For any $z \in \mathbb{C}$, the limit definition of the derivative of $\text{Im}(z)$ is

$$\text{Im}'(z) = \lim_{h \rightarrow 0} \frac{\text{Im}(z+h) - \text{Im}(z)}{h} = \lim_{h \rightarrow 0} \frac{\text{Im } h}{h}.$$

Writing $h = u + iv$, first let $v = 0$, so that $h = u$. Then the limit above (along the horizontal line $h \in \mathbb{R}$) is

$$\lim_{u \rightarrow 0} \frac{\text{Im}(u + 0i)}{u + 0i} = \lim_{u \rightarrow 0} \frac{0}{u} = \lim_{u \rightarrow 0} 0 = 0$$

But if we use $u = 0$, so that $h = iv$, then the limit above (along the vertical line $h \in i\mathbb{R}$) is

$$\lim_{v \rightarrow 0} \frac{\text{Im}(0 + iv)}{0 + iv} = \lim_{v \rightarrow 0} \frac{v}{iv} = \lim_{v \rightarrow 0} -i = -i.$$

Since these two limits disagree, the overall limit defining $\text{Im}'(z)$ diverges. QED Im

Problem 4. II.3, #2. Prove that the functions $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the Cauchy-Riemann equations. Then find a function $f(z)$ (with a simple formula in terms of z) so that $f = u + iv$. (And of course, prove/verify that this formula holds.)

Proof. We compute $\frac{\partial u}{\partial x} = \cos x \sinh y$ and $\frac{\partial v}{\partial y} = \cos x \sinh y = \frac{\partial u}{\partial x}$.

Similarly, We compute $\frac{\partial u}{\partial y} = \sin x \cosh y$ and $\frac{\partial v}{\partial x} = -\sin x \cosh y = -\frac{\partial u}{\partial y}$. QED

By I.8 #1(a) [from HW3], we have $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Thus, using the formulas $\cos(iz) = \cosh(z)$ and $\sin(iz) = i \sinh(z)$ from Section I.8 (page 30), we have $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$.

Therefore, $f = u + iv = \sin x \sinh y + i \cos x \cosh y = i \cos(x + iy) = i \cos(z)$. So $\boxed{f(z) = i \cos(z)}$

Problem 5. II.3, #3. Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \rightarrow \mathbb{C}$. Suppose that both $f(z)$ and its complex conjugate $\bar{f}(z)$ are analytic on D . Prove that f is constant on D .

Proof. Write $f = u + iv$, so that $\bar{f} = u - iv$. Since both are analytic, we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$,

by the first Cauchy-Riemann equation for f and for $\bar{f}$, respectively. Thus, $\frac{\partial u}{\partial x} = 0$ on D .

Similarly, the second Cauchy-Riemann equations for f and for $\bar{f}$ give us $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Thus, $\frac{\partial v}{\partial x} = 0$ on D .

By equation (3.1), then, we have $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0$ on D . By the Theorem on page 49, it follows that f is constant on D . QED