Math 345, Fall 2025

Solutions to Homework #20

Problem 1. VII.2, #9. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}\,dx = \frac{\pi}{2}\left[1-\frac{1}{e^2}\right]$.

Solution. Since $\sin^2 x = \dfrac{1}{2}(1-\cos 2x)$, define $f(z) = \dfrac{1-e^{2iz}}{2(z^2+1)}$, which is analytic except at $z = \pm i$, where it has simple poles. Of those poles, only $z = i$ lies inside the semicircular contour.

The derivative of the denominator of f is $4z$, so by Rule 3, $\text{Res}[f, i] = \dfrac{1-e^{2iz}}{4z}\bigg|_{z=i} = \dfrac{1-e^{-2}}{4i}$.

With Γ_R denoting the semicircular arc portion of the semicircular contour and $zx + iy$ on Γ_R, we have $|e^{2iz}| = |e^{2ix}e^{-2y}| = e^{-2y} \le 1$, so that $|1-e^{2iz}| \le 1+1 = 2$.

Thus, for $R > 1$ and z on Γ_R, we have $|f(z)| = \dfrac{|1-e^{2iz}|}{2|z^2+1|} \le \dfrac{2}{2(|z|^2-1)} = \dfrac{1}{R^2-1}$.

By the ML-estimate, we have $0 \le \left|\displaystyle\int_{\Gamma_R} f(z)\,dz\right| \le \dfrac{\pi R}{R^2-1} \to 0$ as $R \to \infty$. Therefore,

$$\int_{-\infty}^{\infty} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz = \lim_{R\to\infty}\int_{\partial D_R} f(z)\,dz = 2\pi i\,\text{Res}[f,i] = \frac{2\pi i(1-e^{-2})}{4i} = \frac{\pi}{2}\left[1-\frac{1}{e^2}\right].$$

So taking the real part — noting that $\text{Re}\, f(x) = \dfrac{1-\cos 2x}{2(x^2+1)} = \dfrac{\sin^2 x}{x^2+1}$ for $x \in \mathbb{R}$ —

the original integral is $\dfrac{\pi}{2}\left[1-\dfrac{1}{e^2}\right]$, as desired.

Problem 2. VII.4, #1. Let $a \in \mathbb{R}$ with $0 < a < 1$. By integrating around the keyhole contour, show that

$$\int_0^{\infty} \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)}.$$

Solution. Define $f(z) = \dfrac{z^{-a}}{1+z}$ on the slit plane $\mathbb{C} \setminus [0,\infty)$, with a simple pole at $z = -1$, where $z^{-a} = e^{-a\log z}$ for the branch of log given by $\log z = \log|z| + i\arg(z)$ for $0 < \arg z < 2\pi$.

The derivative of the denominator is 1, so by Rule 3, $\text{Res}[f,-1] = e^{-a\log z}\Big|_{z=-1} = e^{-a(0+i\pi)} = e^{-i\pi a}$.

For $0 < \varepsilon < 1 < R$, let Γ_R and γ_ε denote the circles of radius R and ε as in the keyhole contour (with the former traced counterclockwise and the latter traced clockwise).

For z on Γ_R, we have $|z^{-a}| = R^{-a}$, so $|f(z)| = \dfrac{R^{-a}}{|1+z|} \le \dfrac{R^{-a}}{|z|-1} = \dfrac{R^{-a}}{R-1}$.

By the ML-estimate, we have $0 \le \left|\displaystyle\int_{\Gamma_R} f(z)\,dz\right| \le \dfrac{(2\pi R)R^{-a}}{R-1} = \dfrac{2\pi R^{-a}}{1-R^{-1}} \to 0$ as $R \to \infty$, since $a > 0$.

For z on γ_ε, we have $|z^{-a}| = \varepsilon^{-a}$, so $|f(z)| = \dfrac{\varepsilon^{-a}}{|1+z|} \le \varepsilon^{-a}1 - |z| = \dfrac{\varepsilon^{-a}}{1-\varepsilon}$.

By the ML-estimate, we have $0 \le \left|\displaystyle\int_{\gamma_\varepsilon} f(z)\,dz\right| \le \dfrac{(2\pi\varepsilon)\varepsilon^{-a}}{1-\varepsilon} = \dfrac{2\pi\varepsilon^{1-a}}{1-\varepsilon} \to 0$ as $\varepsilon \to 0^+$, since $1-a > 0$.

On the other hand, since the region D enclosed by the keyhole contour ∂D contains the pole $z = -1$, we have $2\pi i e^{-i\pi a} = 2\pi i\,\text{Res}[f,i] = \displaystyle\int_{\Gamma_R} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz + \int_\varepsilon^R f(x)\,dx + \int_R^\varepsilon f(x)\,dx,$

where $\arg x = 0$ in the second-to-last integral, and $\arg x = 2\pi$ in the last integral.
That is, in the second-to-last integral, we have $f(x) = \dfrac{x^{-a}}{1+x}$,
and in the last integral, we have $f(x) = \dfrac{x^{-a} \cdot e^{-2i\pi a}}{1+x}$.
The sum of these last two integrals, then, is $(1 - e^{-2i\pi a}) \displaystyle\int_\varepsilon^R \frac{x^{-a}}{1+x}\,dx$.
Taking the limit as $\varepsilon \to 0^+$ and $R \to \infty$, then, we have $2\pi i e^{-i\pi a} = (1 - e^{-2i\pi a}) \displaystyle\int_0^\infty \frac{x^{-a}}{1+x}\,dx$, so that
$\displaystyle\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{2\pi i e^{-i\pi a}}{1 - e^{-2i\pi a}} = \pi \cdot \frac{2i}{e^{i\pi a} - e^{-i\pi a}} = \pi \cdot \frac{1}{\sin(\pi a)} = \frac{\pi}{\sin(\pi a)}$, as desired.

Problem 3. VII.4, #3. Let $a \in \mathbb{R}$ with $0 < a < 1$. By integrating around the keyhole contour, show that

$$\int_0^\infty \frac{\log x}{x^a(x+1)}\,dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}$$

Solution. Define $f(z) = \dfrac{z^{-a}\log z}{z+1}$ on the slit plane $\mathbb{C} \smallsetminus [0, \infty)$, with a simple pole at $z = -1$, where $z^{-a} = e^{-a \log z}$, and for both appearances of log, we use the branch of log given by
$\log z = \log|z| + i\arg(z)$ for $0 < \arg z < 2\pi$.
The derivative of the denominator is 1, so by Rule 3,
$\operatorname{Res}[f, -1] = e^{-a\log z} \log z \Big|_{z=-1} = e^{-a(0+i\pi)}(i\pi) = i\pi e^{-i\pi a}$.
For $0 < \varepsilon < 1 < R$, let Γ_R and γ_ε denote the circles of radius R and ε as in the keyhole contour (with the former traced counterclockwise and the latter traced clockwise).
For z on Γ_R, we have $|z^{-a}| = R^{-a}$. We also have $|\log z| = |\log R + i \arg z| \le \log R + 2\pi$. Thus, $|f(z)| = \dfrac{R^{-a}|\log z|}{|1+z|} \le \dfrac{R^{-a}(\log R + 2\pi)}{|z| - 1} = \dfrac{R^{-a}(\log R + 2\pi)}{R-1}$.
By the ML-estimate, we have
$0 \le \left| \displaystyle\int_{\Gamma_R} f(z)\,dz \right| \le \dfrac{(2\pi R)R^{-a}(\log R + 2\pi)}{R-1} = \dfrac{2\pi R^{-a}\log R + 4\pi^2 R^{-a}}{1 - R^{-1}} \to 0$ as $R \to \infty$, since $a > 0$,
and since $\displaystyle\lim_{R\to\infty} \frac{\log R}{R^a} = \lim_{R\to\infty} \frac{1/R}{aR^{a-1}} = \lim_{R\to\infty} \frac{1}{a} R^{-a} = 0$ by L'Hôpital's Rule, again because $a > 0$.
For z on γ_ε, we have $|z^{-a}| = \varepsilon^{-a}$. We also have $|\log z| = |\log \varepsilon + i \arg z| \le \log \frac{1}{\varepsilon} + 2\pi$. Thus, $|f(z)| = \dfrac{\varepsilon^{-a}|\log z|}{|1+z|} \le \dfrac{\varepsilon^{-a}(\log\frac{1}{\varepsilon} + 2\pi)}{1-\varepsilon} = \dfrac{\varepsilon^{-a}(\log\frac{1}{\varepsilon} + 2\pi)}{1-\varepsilon}$.
By the ML-estimate, we have
$0 \le \left| \displaystyle\int_{\gamma_\varepsilon} f(z)\,dz \right| \le \dfrac{(2\pi\varepsilon)\varepsilon^{-a}(\log\frac{1}{\varepsilon} + 2\pi)}{1-\varepsilon} = \dfrac{2\pi\varepsilon^{1-a}\log\frac{1}{\varepsilon} + 4\pi^2\varepsilon^{1-a}}{1-\varepsilon} \to 0$ as $\varepsilon \to 0^+$, since $1 - a > 0$.
Here, we have also used the fact that
$\displaystyle\lim_{\varepsilon\to 0^+} \varepsilon^{1-a} \log\frac{1}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{-\log\varepsilon}{\varepsilon^{a-1}} = \lim_{\varepsilon\to 0^+} \frac{-1/\varepsilon}{(a-1)\varepsilon^{a-2}} = \lim_{\varepsilon\to 0^+} \frac{1}{a-1}\varepsilon^{1-a} = 0$ by L'Hôpital's Rule, again because $1 - a > 0$.
On the other hand, since the region D enclosed by the keyhole contour ∂D contains the pole $z = -1$, we have $(2\pi i) \cdot i\pi e^{-i\pi a} = 2\pi i \operatorname{Res}[f, i] = \displaystyle\int_{\Gamma_R} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz + \int_\varepsilon^R f(x)\,dx + \int_R^\varepsilon f(x)\,dx$,

where $\arg x = 0$ in the second-to-last integral, and $\arg x = 2\pi$ in the last integral.

That is, in the second-to-last integral, we have $f(x) = \dfrac{\log x}{x^a(x+1)}$,

and in the last integral, we have $f(x) = \dfrac{e^{-2i\pi a}(\log x + 2\pi i)}{x^a(x+1)} = \dfrac{e^{-2i\pi a}\log x}{x^a(x+1)} + \dfrac{2\pi i e^{-2i\pi a}}{x^a(x+1)}$

The sum of these last two integrals, then, is $(1 - e^{-2i\pi a})\displaystyle\int_\varepsilon^R \frac{\log x}{x^a(x+1)}\,dx - 2\pi i e^{-2\pi i a}\int_\varepsilon^R \frac{x^{-a}}{1+x}\,dx$

Taking the limit as $\varepsilon \to 0^+$ and $R \to \infty$, then, we have

$-2\pi^2 e^{-i\pi a} = (1 - e^{-2i\pi a})\displaystyle\int_0^\infty \frac{\log x}{x^a(x+1)}\,dx - 2\pi i e^{-2\pi i a}\int_0^\infty \frac{x^{-a}}{1+x}\,dx.$

Therefore, using the value of the second integral that we computed in Problem 2, we have

$(1 - e^{-2i\pi a})\displaystyle\int_0^\infty \frac{\log x}{x^a(x+1)}\,dx = -2\pi^2 e^{-i\pi a} + 2\pi i e^{-2\pi i a}\left(\frac{\pi}{\sin(\pi a)}\right) = 2i\pi^2 e^{-i\pi a}\left(i + \frac{e^{-i\pi a}}{\sin(\pi a)}\right)$

Thus, $\displaystyle\int_0^\infty \frac{\log x}{x^a(x+1)}\,dx = \frac{2ie^{-i\pi a}}{1 - e^{-2i\pi a}} \cdot \frac{\pi^2}{\sin(\pi a)}\big(i\sin(\pi a) + e^{-i\pi a}\big)$

$\quad = \dfrac{1}{\sin(\pi a)} \cdot \dfrac{\pi^2}{\sin(\pi a)}\left(\dfrac{e^{i\pi a} + e^{-i\pi a}}{2}\right) = \dfrac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$, as desired.

Problem 4. VII.4 #3, continued, just for fun. Without worrying about switching orders of derivatives and integral signs, "check" the result of the previous problem by differentiating both sides of the formula in Problem 2 (i.e., VII.4 #1) with respect to a, to confirm that we get the formula in Problem 3.

Solution. The derivative of $\dfrac{x^{-a}}{1+x} = \dfrac{e^{-a\log x}}{1+x}$ with respect to a is $\dfrac{-\log x \cdot e^{-a\log x}}{1+x} = -\dfrac{\log x}{x^a(x+1)}$.
That is, the derivative of the integrand in Problem 2 is the negative of the integrand in Problem 3.

The derivative of $\dfrac{\pi}{\sin(\pi a)}$ with respect to a is $\dfrac{-\pi\cos(\pi a)\cdot\pi}{(\sin(\pi a))^2} = -\dfrac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}$.
That is, the derivative of the right side in Problem 2 is the negative of the right side in Problem 3.

Taking negatives, then, the derivative (with respect to a) of the formula in Problem 2 gives the formula in Problem 3.