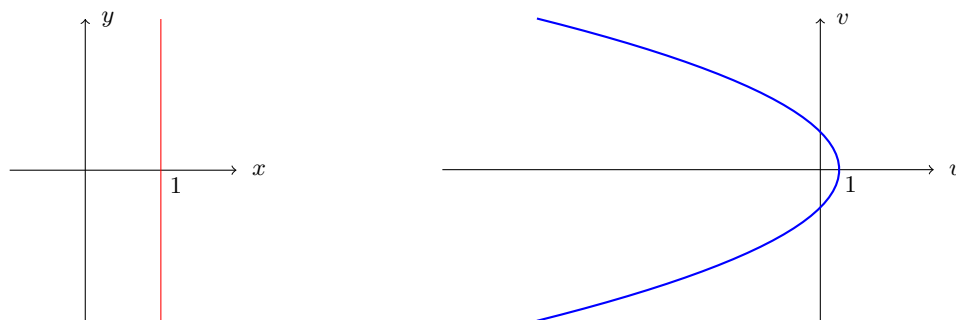Math 345, Fall 2025

Solutions to Selected Homework Problems, HW #2

Problem 1: **I.4, #1(b)** Sketch the curve $x = 1$ in the z-plane, and then sketch its image in the w-plane under $w = z^2$.

Solution. Write $z = 1+iy$ and $w = u+iv$. Then since $w = z^2$, we have $u+iv = (1+iy)^2 = (1-y^2)+2yi$.

That is, we have $u = 1 - y^2$ and $v = 2y$. Thus, $y = \dfrac{v}{2}$, so we have $u = 1 - \dfrac{v^2}{4}$.

So here are the graphs of $x = 1$ and its image $u = 1 - \dfrac{v^2}{4}$:

Problem 2: **I.4, #1(f)** Sketch the curve $y = 1/x$ (for $x \neq 0$) in the z-plane, and then sketch its image in the w-plane under $w = z^2$.

Solution. Write $z = x + \dfrac{i}{x}$ with $x \neq 0$, and $w = u + iv$. Then since $w = z^2$, we have

$$u + iv = \left(x + \frac{i}{x}\right)^2 = x^2 - \frac{1}{x^2} + 2i.$$ That is, we have $u = x^2 - x^{-2}$ and $v = 2$.

Note that the (real) function $u(x) = x^2 - x^{-2}$ is continuous on the domain $x \in (0, \infty)$ and has $\lim\limits_{x \to \infty} u(x) = +\infty$ and $\lim\limits_{x \to 0^+} u(x) = -\infty$. Thus, u must map $(0, \infty)$ onto the full real line $(-\infty, \infty)$.

Similarly, u also maps $(-\infty, 0)$ onto the full real line $(-\infty, \infty)$. So the original curve $z = x + \dfrac{i}{x}$ with $x \neq 0$ covers the line $\text{Im}(w) = 2$ twice.

So here are the graphs of the $z = x + \dfrac{i}{x}$ and its image curve, $\text{Im}(w) = 2$.
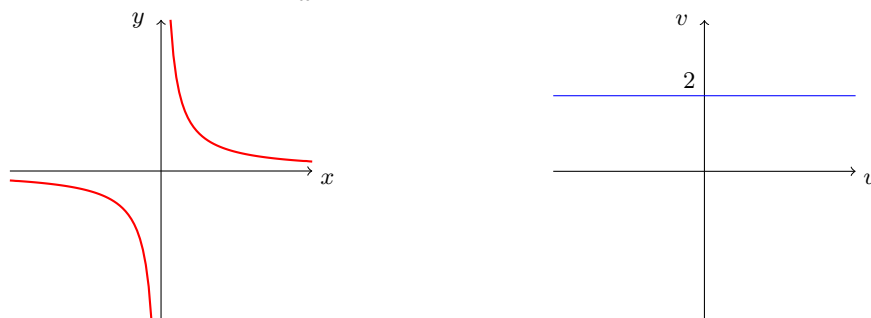
Problem 3: I.4, #2(b): Sketch the image of $x = 1$ under both branches of $w = \sqrt{z}$, in different colors.

Solution. The union of the two desired curves must be the set of all complex numbers $w = u + iv$ for which $\text{Re}(w^2) = 1$. Since $w^2 = (u^2 - v^2) + 2uvi$, this means the two curves together form the hyperbola $u^2 - v^2 = 1$.

This hyperbola consists of two separate arcs, one with $u \geq 1$ and one with $u \leq -1$. Since the image of the principal branch of $w = \sqrt{z}$ lies in the half-plane $\operatorname{Re} w > 0$, and the other branch's image lies in the half-plane $\operatorname{Re} w < 0$, each of the arcs must be one branch. So here they are, with the image of the principal branch in blue and the other in red.
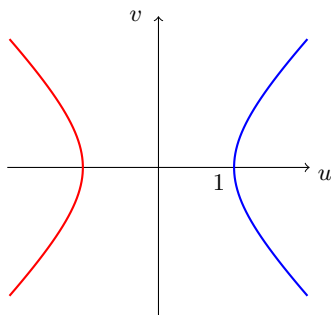
Problem 4: **I.4, #3(a)** Give a brief description of the function $w = z^3$. (Describe what happens to a ray from the origin in the z-plane, and to a circle centered at the origin.)

Solution. Writing $z = re^{i\theta}$ in polar, we have $w = r^3 e^{i(3\theta)}$, so that $|w| = |z|^3$ and $\arg w = 3 \arg z$.

Since $\arg w = 3 \arg z$, the ray from the original at angle θ_0 to the positive real axis is mapped to the ray at angle $3\theta_0$. (And because $|w| = |z|^3$, if z traverses this ray starting at the origin, then it starts moving slowing and then with increasing speed.)

Since $|w| = |z|^3$, the circle of radius r_0 centered at the origin is mapped to the circle of radius r^3. (And because $\arg w = 3 \arg z$, this mapping is 3-to-1; one loop around the z-circle traces out three loops around the w-circle.)

Problem 5: **I.4, #3(b)** Make branch cuts and define explicitly three branches of the inverse mapping of $w = z^3$.

Solution. As in the discussion of $\sqrt{z}$ in Section I.4, make a branch cut in the w-plane by removing the branch cut $(-\infty, 0]$. So each $w \in \mathbb{C} \smallsetminus (-\infty, 0]$ has a well defined Argument $\operatorname{Arg} w \in (-\pi, \pi)$.

For each $w \in \mathbb{C} \smallsetminus (-\infty, 0]$, write $w = \rho e^{i\varphi}$, and define
$$f_1(w) = \rho^{1/3} e^{i\varphi/3}, \qquad f_2(w) = \rho^{1/3} e^{i(\varphi+2\pi)/3}, \qquad f_3(w) = \rho^{1/3} e^{i(\varphi+4\pi)/3}.$$
Then for each $j = 1, 2, 3$, we have
$$(f_j(w))^3 = \rho e^{i\varphi} = w,$$
so that each f_j is a branch of the inverse mapping to $w = z^3$.

Incidentally, note that the three branches are definitely different, because for any such $w = \rho e^{i\varphi}$, we have $-\pi/3 < \varphi/3 < \pi/3$, and hence
$$\operatorname{Arg}(f_1(w)) \in (-\pi/3, \pi/3), \qquad \operatorname{Arg}(f_2(w)) \in (\pi/3, \pi), \qquad \operatorname{Arg}(f_3(w)) \in (-\pi, -\pi/3),$$
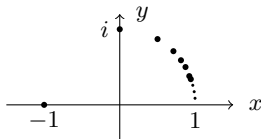which describe three disjoint sectors of the z-plane.

Problem 6: **I.5, #1(b,d)** Calculate and plot e^z for the following points z: (b): $\pi i + 1$ (d): $37\pi i$

Solution. $e^{\pi i + 1} = e^{\pi i} e^1 = (-1)(e) = -e$, and $e^{37\pi i} = e^{\pi i} e^{36\pi i} = (-1)(1) = -1$, like so:

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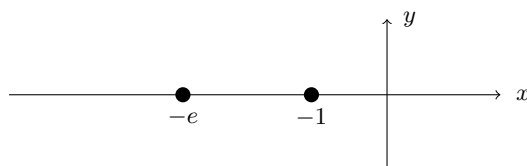
Problem 7: **I.5, #1(e)** Calculate and plot e^z for $z = \pi i / m$ for $m = 1, 2, 3, \dots$

Solution. For $m = 1$ we have $e^{\pi i} = -1$, for $m = 2$ we have $e^{\pi i/2} = i$, and for $m \geq 3$ we get the point of modulus 1 at angle π/m in the first quadrant. That is:

where the points lie on the circle $|z| = 1$ and approach the point 1 (but never get there).

Problem 8: **I.5, #2(c)** Sketch the region $0 < x < 1$, $0 < y < \pi/4$ and its imagine under $w = e^z$. Indicate the images of horizontal and vertical lines.

Solution. Write $z = x + iy$ and $w = se^{i\phi}$. Then $se^{i\phi} = w = e^z = e^x e^{iy}$, so we have $s = e^x$ and $\phi = y$ (with ϕ defined only up to adding integer multiples of 2π).
Since $0 < x < 1$, we have $1 < e^x < e$, so $1 < |w| < e$. In addition, each vertical line $\operatorname{Re}(z) = a$ becomes $s = e^a$, i.e., the circle $|w| = e^a$.
Since $0 < y < \pi/4$, we have $0 < \phi < \pi/4$, so $0 < \operatorname{Arg}(w) < \pi/4$. In addition, each horizontal line $\operatorname{Im}(z) = b$ becomes $\phi = b$, i.e., the ray $\operatorname{Arg}(w) = b$.
So here is the rectangle and its image:

Problem 9: **I.5, #3**. Show that $e^{\bar{z}} = \overline{e^z}$.

Proof. Given $z \in \mathbb{C}$, write $z = x + iy$. Then $\bar{z} = x - iy$, so
$$e^{\bar{z}} = e^x e^{-iy} = e^x \big(\cos y + i \sin(-y)\big) = e^x \big(\cos y - i \sin y\big) = \overline{e^x \big(\cos y + i \sin y\big)} = \overline{e^x e^{iy}} = \overline{e^z} \qquad \text{QED}$$