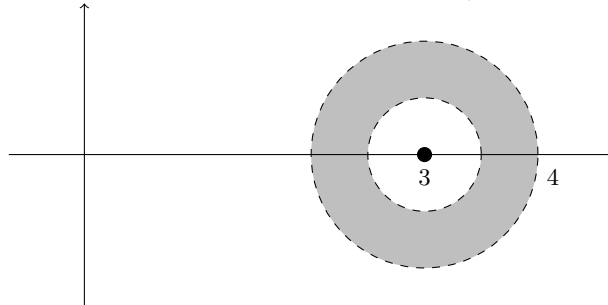Math 345, Fall 2025

Solutions to Selected Homework Problems, HW #1

Problem 1: **I.1, #1(b)** Identify and sketch the set of points $1 < |2z - 6| < 2$

Solution. Since $|2z - 6| = |2| \cdot |z - 3| = 2|z - 3|$, this set is given by $\frac{1}{2} < |z - 3| < 1$. That is, it is the set of points $z \in \mathbb{C}$ whose distance from 3 is strictly between $1/2$ and 1. So it's this open annulus:


Problem 2: **I.1, #1(e)** Identify and sketch the set of points $|z - 1| < |z|$.

Solution. Since $|z - 1|$ is the distance from z to 1, and $|z|$ is the distance from z to 0, the set where $|z - 1| < |z|$ is the set of points $z \in \mathbb{C}$ which are closer to 1 than to 0. The set of points where these two distances are the same is the vertical line $\operatorname{Re} z = \frac{1}{2}$, because on this line, the distances from z to each of 0 and 1 are the same, since the three points form an isosceles triangle:
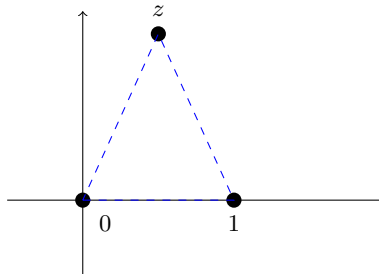
So the desired set is the set of points strictly to the *right* of that vertical line, which is this open half-plane:
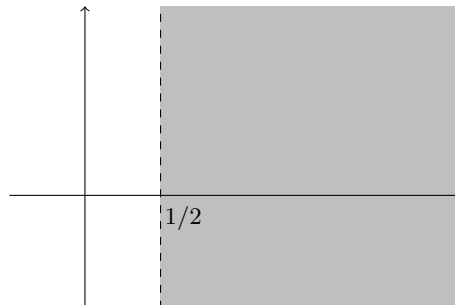
Alternative argument to see that the region is $\operatorname{Re} z > 1/2$. Squaring both sides of the (real) inequality $|z - 1| < |z|$ gives $|z - 1|^2 < |z|^2$. Writing $z = x + iy$, this inequality is just $(x - 1)^2 + y^2 < x^2 + y^2$, which expands to $-2x + 1 < 0$, i.e., $1 < 2x$, or equivalently $x > 1/2$. That is, $\operatorname{Re} z > 1/2$.

Problem 3: **I.1, #2(a,b)** Verify the identities $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\,\bar{w}$

Proof. Given $z, w \in \mathbb{C}$, write $z = x + iy$ and $w = u + iv$ with $x, y, u, v \in \mathbb{R}$.

(a): Then $\overline{z+w} = \overline{(x+u)+i(y+v)} = (x+u)-i(y+v) = (x-iy)+(u-iv) = \bar{z}+\bar{w}$ QED (a)

(b): We have:

$\overline{zw} = \overline{(xu-yv)+i(xv+yu)} = (xu-yv)-i(xv+yu) = (x-iy)(u-iv) = \bar{z}\,\bar{w}$ QED (b)

Problem 4: **I.1, #5, first half**: Prove (for all $z \in \mathbb{C}$) that $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

Proof. Given $z \in \mathbb{C}$, write $z = x+iy$. Then

$|\operatorname{Re} z| = |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$ and $|\operatorname{Im} z| = |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} = |z|.$ QED

Problem 5: **I.1, #5, second half**: Prove (for all $z, w \in \mathbb{C}$) that $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})$. Then use this to prove the triangle inequality: $|z+w| \le |z| + |w|$.

Proof. Given $z, w \in \mathbb{C}$, let $c = z\bar{w} \in \mathbb{C}$. Then

$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + c + \bar{c},$

since $\bar{c} = \overline{z\bar{w}} = \bar{z}\overline{\bar{w}} = \bar{z}w$.

Writing $c = a + bi$ with $a, b \in i$, we have $c + \bar{c} = (a+bi) + (a-bi) = 2a = 2\operatorname{Re}(c)$.

Thus, $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(c) = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})$.

Finally, to prove the triangle inequality, the previous identity gives

$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}) \le |z|^2 + |w|^2 + 2|\operatorname{Re}(z\bar{w})| \le |z|^2 + |w|^2 + 2|z\bar{w}|$
$= |z|^2 + |w|^2 + 2|z||\bar{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z|+|w|)^2.$

All quantities here are nonnegative real numbers, so taking square roots gives $|z+w| \le |z|+|w|$. QED

Alternative proof of first identity. Given $z, w \in \mathbb{C}$, write $z = x+iy$ and $w = u+iv$ with $x, y, u, v \in \mathbb{R}$. Then

$|z+w|^2 = |(x+u)+i(y+v)|^2 = (x+u)^2 + (y+v)^2 = x^2 + 2xu + u^2 + y^2 + 2yv + v^2$
$= (x^2+y^2) + (u^2+v^2) + 2(xu+yv) = |z|^2 + |w|^2 + 2\operatorname{Re}\big((xu+yv)+i(yu-xv)\big)$
$= |z|^2 + |w|^2 + 2\operatorname{Re}\big((x+iy)(u-iv)\big) = |z|^2 + |w|^2 + 2\operatorname{Re}\big(z\bar{w}\big)$

Problem 6: **I.2, #1(b,e)**: Express all values of $\sqrt{i-1}$ and $(-8)^{-1/3}$ in both polar and cartesian, and plot them.
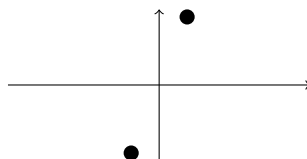
Solution, (b). We have $|i-1| = \sqrt{(-1)^2+1^2} = \sqrt{2}$, and $\operatorname{Arg}(i-1) = 3\pi/4$, as we see here:
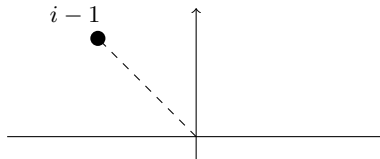
Thus, the two square roots of $i-1$ are of the form $re^{i\theta}$ where $r = \sqrt[4]{2}$ and $\theta = \dfrac{3}{8}\pi + \dfrac{2\pi}{2}n$ for $n = 0, 1$. [To make the numbers smaller, let's actually subtract 2π from the second θ to get $-5\pi/8$.]

So in polar, the two roots are $\boxed{\sqrt[4]{2}e^{3\pi i/8} \text{ and } \sqrt[4]{2}e^{-5\pi i/8}}$

which in cartesian are $\boxed{\sqrt[4]{2}\cos(3\pi/8) + i\sqrt[4]{2}\sin(3\pi/8)}$ and $\boxed{\sqrt[4]{2}\cos(5\pi/8) - i\sqrt[4]{2}\sin(5\pi/8)}$, like so:


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Note 1: When I wrote the cartesian answers above, I used the facts that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Note 2: the second root is, of course, simply the negative of the first, since it is $e^{i\pi}$ times the first.
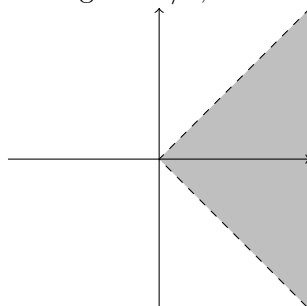
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Solution, (e). We have $|-8| = 8$, and $\text{Arg}(-8) = \pi$, so the three cube roots of -8 are of the form $re^{i\theta}$ where $r = \sqrt[3]{8} = 2$ and $\theta = \frac{1}{3}\pi + \frac{2\pi}{3}n$ for $n = 0, 1, 2$. That is, $\theta = \frac{\pi}{3}\pi, \pi, -\frac{\pi}{3}$, where we subtracted 2π from the third θ.

So in polar, the three roots are $\boxed{2e^{\pi i/3},\ 2e^{\pi i},\ 2e^{-\pi i/3}}$, like so:
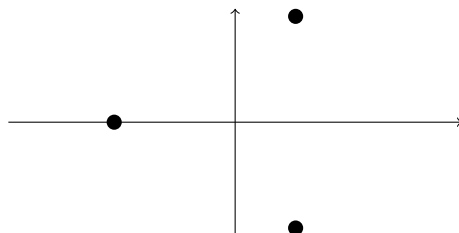
Problem 7: **I.2, #2(a)**: Sketch the set of points $z \in \mathbb{C}$ for which $|\arg z| < \pi/4$.

Solution. This condition means $-\pi/4 < \arg z < \pi/4$, which looks like this:

Problem 8: **I.2, #4**: For which integers $n \geq 1$ is i an n-th root of unity?

Solution/Proof. We are being asked for the set of integers $n \geq 1$ for which $i^n = 1$. We claim that this set is precisely $\boxed{\text{those integers } n \geq 1 \text{ that are divisble by 4}}$

To see this, writing an arbitrary integer $n \geq 1$ as $n = 4k + j$, where k is an integer and $j \in \{0, 1, 2, 3\}$. We must show that $i^n = 1$ if and only if $j = 0$.

If $j = 0$, then $i^n = i^{4k} = (i^4)^k = 1^k = 1$, as desired.

Conversely, if $j \in \{1, 2, 3\}$, then $i^n = i^{4k} \cdot i^j = i^j \neq 1$, since $i^1 = i$ and $i^2 = -1$ and $i^3 = -i$.   QED

Problem 9: **I.2, #8, first part**: Prove that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\cos\theta\sin\theta$ using DeMoivre's formulae.

Proof. DeMoivre for $n = 2$ says, for all $\theta \in \mathbb{R}$

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

Taking real parts of both sides gives $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

Taking imaginary parts gives $\sin(2\theta) = 2\cos\theta\sin\theta$, as desired.   QED

Problem 10: **I.2, #8, second part**: Find and prove formulae for $\cos(4\theta)$ and $\sin(4\theta)$.

Solution/Proof. For $n = 4$, DeMoivre gives

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Taking real parts gives $\boxed{\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Taking imaginary parts gives $\boxed{\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

Alternative derivation of last identities. Applying the 2θ identities to 2θ itself, we have

$\cos(4\theta) = \cos\big(2(2\theta)\big) = \cos^2(2\theta) - \sin^2(2\theta) = \big(\cos^2(\theta) - \sin^2(\theta)\big)^2 - \big(2\cos\theta\sin\theta\big)^2$
$\quad = \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta - 4\cos^2\theta\sin^2\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

and $\sin(4\theta) = \sin\big(2(2\theta)\big) = 2\cos(2\theta)\sin(2\theta) = 2\big(\cos^2(\theta) - \sin^2(\theta)\big) \cdot \big(2\cos\theta\sin\theta\big)$
$\quad = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$