Math 345, Fall 2024 Professor Rob Benedetto

Final Exam, Take-Home
Due **Wednesday, December 18** in Gradescope by **11:59 pm ET**
(You are welcome to submit it before that, if you want.)

Instructions: **Answers must be written neatly and legibly, and tagged to the correct problem numbers on Gradescope.**

You must fully justify your answers. (So on computational problems, show all steps and explain your thinking; on proof problems, give rigorous proofs.)

You may quote theorems from class or from the sections of the book that we covered, as well as standard theorems from single variable and multivariable calculus (Math 111, 121, 211). You may also quote the results of any assigned (**non-challenge**) homework problems, or from any of the (**non-Bonus**) problems from the two midterms, whether or not you correctly solved those problems yourself.

However, you must clearly verify all the hypotheses of any theorem you use, and you must clearly name or reference any such theorem. (E.g., "By the Theorem on page 50..., or "By Green's Theorem...," or "By Problem 4 of Exam 1....") If you are not sure whether or not some argument or statement requires further justification, please ask me about it.

You may use Gamelin (Chapters I–VII), your notes, your own old homework, and any materials from the **course** websites, including handouts, problem solutions, and videos from the course. You may also consult a textbook on single and/or multivariable calculus (like Stewart's *Calculus*), provided it does not cover material beyond standard multivariable calculus.

You may NOT use other books, online information, AI, calculators, or any other outside sources.
You also may NOT discuss the problems with anyone other than me.

But you **should** feel free to talk to me about anything on the exam.

There are 8 problems, totalling 200 points, plus an optional bonus problem.

1. (**20 points**). Let $f(z) = \dfrac{\mathrm{Log}(1-z)}{z^4(z-1)^3}$. Show that f has a pole at $z=0$. Compute the order of the pole and the principal part of f at $z=0$.

2. (**20 points**). Find all Laurent series centered at 0 for $g(z) = \dfrac{1}{z^3(z^2-6)}$. Specify the domains on which each is equal to g.

3. (**25 points**). Given $a \in \mathbb{C}$ with $|a| \neq 1$, compute $\displaystyle\int_{|z|=1} \frac{z+1}{(z-a)^2 \sin z}\, dz$.

(*Note: there are three cases to consider.*)

4. (**15 points**). Let $D \subseteq \mathbb{C}$ be a domain, and let $f, g : D \to \mathbb{C}$ be analytic functions. Suppose that $f(z) \cdot g(z) = 0$ for all $z \in D$. Prove that either $f(z) = 0$ for all $z \in D$ or $g(z) = 0$ for all $z \in D$.

(*Suggestion*: What can you say about (non)isolated zeros?)

5. (**15 points**). Let $\{a_k\}_{k\geq 0}$ be a sequence in $\mathbb{C}\setminus\{0\}$. The *infinite product* $\prod_{k=0}^{\infty} a_k$ is defined to be $\boxed{\prod_{k=0}^{\infty} a_k = \lim_{N\to\infty}\prod_{k=0}^{N} a_k}$ if the limit converges **and is nonzero**.

(Of course, $\prod_{k=0}^{N} a_k$ means the product $a_0 \cdot a_1 \cdot \cdots \cdot a_N$.)

Prove that if $\prod_{k=0}^{\infty} a_k$ converges (and is nonzero), then $\lim_{k\to\infty} a_k = 1$.

6. (**30 points**). Use the semicircular contour to compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\cos x}{x^4+4}\,dx$.

7. (**35 points**). Given real numbers a, b, c with $-1 < a < 1$ and $a \neq 0$, and $b > c > 0$, use the keyhole contour to compute $\displaystyle\int_0^{\infty} \frac{x^a}{(x+b)(x+c)}\,dx$.

8. (**40 points**). In this problem, you'll compute $\displaystyle\sum_{k=1}^{\infty}\binom{2k}{k+1}\frac{1}{6^k}$. (Recall that $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$, which is a certain coefficient in the expansion of $(a+b)^n$.)

(a) Show that for every integer $k \geq 1$, $\displaystyle\int_{|z|=1} \frac{(1+z)^{2k}}{z^k}\,dz = 2\pi i\binom{2k}{k+1}$.

(b) Evaluate $\displaystyle\int_{|z|=1} \frac{z\,dz}{z^2-4z+1}$. (*Note*: make sure you know where the poles are!)

(c) Prove that $\displaystyle\sum_{k=0}^{\infty} \frac{(1+z)^{2k}}{(6z)^k}$ converges uniformly to $\dfrac{-6z}{z^2-4z+1}$ on the circle $|z| = 1$.
(*Hint*: Geometric series. Don't forget to show uniformity!)

(d) Use parts (a)–(c) to evaluate $\displaystyle\sum_{k=1}^{\infty}\binom{2k}{k+1}\frac{1}{6^k}$.

OPTIONAL BONUS (3 points). Compute $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+25}$, by **proving** and then applying:

Theorem. Let $f(z) = \dfrac{P(z)}{Q(z)}$ be a rational function with $\deg Q \geq 2+\deg P$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be the distinct zeros of Q. (That is, any of the α_i may be a multiple root of Q, but we have $\alpha_i \neq \alpha_j$ for all $i \neq j$.) Then

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq \text{any } \alpha_j}} f(k) = -\pi\sum_{j=1}^{n}\Big(\operatorname{Res}\big[f(z)\cot(\pi z), \alpha_j\big]\Big).$$

(*Suggestion*: Integrate $\pi f(z)\cot(\pi z)$ around a large square centered at the origin, and of side length $2N+1$. Then take the limit as the integer N goes to ∞.)