

## Solutions to Practice Problems for Midterm Exam 2

$$1.(a) \int_{|z|=6} \frac{\cos(\pi z)}{(z-1)^3} dz$$

**Answer.** Let  $D = D(0, 6)$ , and  $f(z) = \cos(\pi z)$ , which is analytic everywhere, and in particular on  $\overline{D}$ . We have  $f'(z) = -\pi \sin \pi z$  and hence  $f''(z) = -\pi^2 \cos \pi z$ . Since  $1 \in D$ , by CDF [Cauchy Differentiation Formula] the integral is  $\int_{\partial D} \frac{f(z)}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1) = \pi i \cdot (-\pi^2 \cos \pi) = \pi^3 i$ .

$$1.(b) \int_{|z|=2} \frac{e^{i\pi z/2}}{z(z+1)(z+3)} dz$$

**Answer.** First, let  $D = D(0, 2) \setminus [\overline{D}(0, 1/3) \cup \overline{D}(-1, 1/3)]$ , and let  $f(z)$  be the integrand of this integral. Then  $f$  is analytic on  $\overline{D}$ , and hence  $\int_{\partial D} f(z) dz = 0$ . Since  $\partial D$  consists of three circles, with the inner ones traced in the negative direction, we have

$$\int_{|z|=2} f(z) dz = \int_{|z|=1/3} f(z) dz + \int_{|z+1|=1/3} f(z) dz.$$

For the first of these two integrals, use  $g(z) = e^{i\pi z/2}/[(z+1)(z+3)]$ , which is analytic on  $\overline{D}(0, 1/3)$ . By CIF [Cauchy Integral Formula], this first integral is  $2\pi i g(0) = 2\pi i/3$ .

For the second, use  $h(z) = e^{i\pi z/2}/[z(z+3)]$ , which is analytic on  $\overline{D}(-1, 1/3)$ . By CIF [Cauchy Integral Formula], this second integral is  $2\pi i h(-1) = 2\pi i e^{-i\pi/2}(-1/2) = -\pi$ .

So the original integral is  $-\pi + 2\pi i/3$ .

$$1.(c) \int_{|z-3|=2} \frac{\text{Log } z}{z^2(z-4)^2} dz$$

**Answer.** Let  $D = D(3, 2)$ , and  $f(z) = \text{Log } z/z^2$ , which is analytic on  $\mathbb{C} \setminus (-\infty, 0]$  and hence on  $\overline{D}$ . We have  $f'(z) = [(1/z)z^2 - 2z \text{Log } z]/z^4 = (1 - 2 \text{Log } z)/z^3$ . Since  $4 \in D$ , by CDF the integral is  $\int_{\partial D} \frac{f(z)}{(z-4)^2} dz = \frac{2\pi i}{1!} f'(4) = 2\pi i \left( \frac{1 - 2 \log 4}{4^3} \right) = \frac{\pi i(1 - 4 \log 2)}{32}$ . [And yes,  $2 \log 4 = 4 \log 2$ , although you can leave it as  $2 \log 4$  if you want.]

$$1.(d) \int_{|z|=4} \frac{e^{5z}}{(z-\pi i)^3} dz$$

**Answer.** Let  $D = D(0, 4)$ , and  $f(z) = e^{5z}$ , which is analytic on  $\mathbb{C}$  and hence on  $\overline{D}$ . We have  $f'(z) = 5e^{5z}$  and  $f''(z) = 25e^{5z}$ . Since  $\pi i \in D$ , by CDF the integral is

$$\int_{\partial D} \frac{f(z)}{(z-\pi i)^3} dz = \frac{2\pi i}{2!} f''(\pi i) = \pi i (25e^{5\pi i}) = -25\pi i.$$

$$1.(e) \int_{|z|=\pi} \frac{e^{5z}}{(z-4)^4} dz$$

**Answer.** Let  $D = D(0, \pi)$ , and  $f(z) = e^{5z}/(z-4)^4$ , which is analytic on  $\mathbb{C} \setminus \{4\}$  and hence on  $\overline{D}$ . [Note:  $4 \notin \overline{D}$ !!!] So by Cauchy's Theorem, the integral is 0.

$$1.(f) \int_{|z-5|=4} \frac{(z-3) \sin z}{z^3(z-6)(z-8)} dz$$

**Answer.** Let  $D = D(5, 4) \setminus [\overline{D}(6, 1/2) \cup \overline{D}(8, 1/2)]$ , and let  $f(z)$  be the integrand of this integral. Then  $f$  is analytic on  $\overline{D}$ , and hence  $\int_{\partial D} f(z) dz = 0$ . Since  $\partial D$  consists of three circles, with the inner ones traced in the negative direction, we have

$$\int_{|z-5|=4} f(z) dz = \int_{|z-6|=1/2} f(z) dz + \int_{|z-8|=1/2} f(z) dz.$$

For the first of these two integrals, use  $g(z) = (z-3) \sin z / [z^3(z-8)]$ , which is analytic on  $\overline{D}(6, 1/2)$ . By CIF [Cauchy Integral Formula], this first integral is  $2\pi i g(6) = 2\pi i (3 \sin 6) / (6^3 \cdot (-2)) = -i\pi \sin 6 / 72$ . For the second, use  $h(z) = (z-3) \sin z / [z^3(z-6)]$ , which is analytic on  $\overline{D}(8, 1/2)$ . By CIF [Cauchy Integral Formula], this second integral is  $2\pi i h(8) = 2\pi i (5 \sin 8) / (8^3 \cdot 2) = 5i\pi \sin 8 / 512$ .

So the original integral is  $i\pi \left( \frac{5 \sin 8}{512} - \frac{\sin 6}{72} \right)$ .

2. For each of the following functions, find its full power series expansion about  $z = 0$ , as well as the radius of convergence of this power series.

**Answers.** (a)  $f(z) = z \operatorname{Log}(z+2)$ :

First, we note that  $\frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1 - (-z/2)} = \sum_{k \geq 0} \frac{(-1)^k}{2^{k+1}} z^k$ , with radius of convergence  $R = 2$ , since  $1/(z+2)$  is analytic on  $\mathbb{C} \setminus \{-2\}$  but blows up at  $-2$ , and  $|-2 - 0| = 2$ .

Antidifferentiating,  $\operatorname{Log}(z+2) = C + \sum_{k \geq 0} \frac{(-1)^k}{2^{k+1}(k+1)} z^{k+1} = C + \sum_{k \geq 1} \frac{(-1)^{k-1}}{k \cdot 2^k} z^k$  for some constant  $C$ ,

with the same radius of convergence  $R = 2$ .

Plugging  $z = 0$  into both sides, we have  $C = \operatorname{Log} 2$ . Finally, multiplying by  $z$  (which does not change  $R$ ), we have  $f(z) = z \log 2 + \sum_{k \geq 2} \frac{(-1)^k}{(k-1) \cdot 2^{k-1}} z^k$ , with radius of convergence  $R = 2$ .

(b)  $g(z) = \frac{z^2}{(z^5 - 4)^3}$ :

First, we note that  $\frac{1}{z-4} = \frac{-1}{4} \cdot \frac{1}{1 - (z/4)} = \sum_{k \geq 0} \frac{-1}{4^{k+1}} z^k$ , with radius of convergence  $R = 4$ , since  $1/(z-4)$  is analytic on  $\mathbb{C} \setminus \{4\}$  but blows up at  $4$ , and  $|4 - 0| = 4$ .

Differentiating,  $\frac{-1}{(z-4)^2} = \sum_{k \geq 0} \frac{-k}{4^{k+1}} z^{k-1}$ .

Differentiating again,  $\frac{2}{(z-4)^3} = \sum_{k \geq 0} \frac{-k(k-1)}{4^{k+1}} z^{k-2} = \sum_{k \geq 0} \frac{-(k+1)(k+2)}{4^{k+3}} z^k$ , with the same radius of convergence  $R = 4$ .

Composing with  $z^5$ , we have  $\frac{2}{(z^5 - 4)^3} = \sum_{k \geq 0} \frac{-(k+1)(k+2)}{4^{k+3}} z^{5k}$ , with radius of convergence  $R = \sqrt[5]{4}$ .

Finally, multiplying by  $z^2/2$ , we have  $g(z) = \sum_{k \geq 0} \frac{-(k+1)(k+2)}{2 \cdot 4^{k+3}} z^{5k+2}$ , with radius of convergence  $R = \sqrt[5]{4}$ .

3. Let  $h(z) = (z^2 + 1) \sin(2z^3)$ . Compute the following derivatives:  $h^{(15)}(0)$ ,  $h^{(16)}(0)$ , and  $h^{(17)}(0)$ .

**Answer.** Since  $\sin z = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$ , we have  $\sin(2z^3) = \sum_{k \geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+3}$ , and hence also

$$z^2 \sin(2z^3) = \sum_{k \geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+5}. \text{ Adding gives } h(z) = \sum_{k \geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+3} + \sum_{k \geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+5}.$$

Note that the first sum uses only powers of  $z$  that are  $3 \pmod{6}$ , while the second uses only powers

$$\text{that are } 5 \pmod{6}. \text{ That is, } h(z) = \sum_{n \geq 0} a_n z^n, \text{ where } a_n = \begin{cases} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} & \text{if } n = 6k+3 \text{ for some } k, \\ \frac{(-1)^k 2^{2k+1}}{(2k+1)!} & \text{if } n = 6k+5 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

So since  $15 = 6k+3$  for  $k=2$ , we have  $h^{(15)}(0) = 15!a_{15} = \frac{15! \cdot 2^5}{5!}$ .

And since 16 is neither 3 or 5  $\pmod{6}$ , we have  $h^{(16)}(0) = 16!a_{16} = 0$ .

Finally, since  $17 = 6k+5$  for  $k=2$ , we have  $h^{(17)}(0) = 17!a_{17} = \frac{17! \cdot 2^5}{5!}$ .

4. (a) Find the power series centered at  $z=0$  of these two functions:

$$g(z) = ze^{(z^2)} \quad \text{and} \quad h(z) = \cos(2z).$$

(b) Consider the power series  $\sum_{k=0}^{\infty} a_k z^k$  centered at  $z=0$  for  $f(z) = \frac{ze^{(z^2)}}{\cos(2z)}$ .

Use part (a) to compute  $a_k$  for each of  $k=0, \dots, 6$ .

(c) For  $f(z)$  as in part (b), compute  $f^{(5)}(0)$ .

(d) What is the radius of convergence of the power series in part (b)?

(Briefly) explain why.

**Answers.** (a): Since  $e^z = \sum_{k \geq 0} \frac{z^k}{k!}$ , we have  $e^{z^2} = \sum_{k \geq 0} \frac{z^{2k}}{k!}$ , and hence  $g(z) = ze^{z^2} = \sum_{k \geq 0} \frac{z^{2k+1}}{k!}$ .

Since  $\cos z = \sum_{k \geq 0} \frac{(-1)^k z^{2k}}{(2k)!}$ , we have  $h(z) = \cos 2z = \sum_{k \geq 0} \frac{(-1)^k z^{2k}}{(2k)!}$ .

(b): Writing each of the power series of part (a) up to the  $z^6$  terms, we have

$$g(z) = z + z^3 + \frac{z^5}{2} + O(z^7) \quad \text{and} \quad h(z) = 1 - 2z^2 + \frac{2}{3}z^4 - \frac{4}{45}z^6 + O(z^7).$$

Thus,

$$\begin{aligned} \frac{1}{h(z)} &= \frac{1}{1 - (2z^2 - \frac{2}{3}z^4 + \frac{4}{45}z^6 + O(z^7))} = 1 + (2z^2 - \frac{2}{3}z^4 + \frac{4}{45}z^6) + (2z^2 - \frac{2}{3}z^4)^2 + (2z^2)^3 + O(z^7) \\ &= 1 + 2z^2 + \left(-\frac{2}{3} + 4\right)z^4 + \left(\frac{4}{45} - \frac{8}{3} + 8\right)z^6 + O(z^7) = 1 + 2z^2 + \frac{10}{3}z^4 + \frac{244}{45}z^6 + O(z^7) \end{aligned}$$

Multiplying, then, we have

$$f(z) = \frac{g(z)}{h(z)} = z + (1+2)z^3 + \left(\frac{1}{2} + 2 + \frac{10}{3}\right)z^5 + O(z^7) = z + 3z^3 + \frac{35}{6}z^5 + O(z^7).$$

That is,  $a_0 = a_2 = a_4 = a_6 = 0$ ,  $a_1 = 1$ ,  $a_3 = 3$ , and  $a_5 = 35/6$ .

(c): We have  $f^{(5)}(0) = 5! \cdot a_5 = \frac{120 \cdot 35}{6} = 20 \cdot 35 = 700$

(d):  $f = g/h$  is analytic everywhere that  $h(z) \neq 0$ . In particular, the closest points to 0 at which  $\cos(2z) = 0$  are where  $2z = \pm\pi/2$ , i.e.,  $z = \pm\pi/4$ . In particular,  $f$  is analytic on  $D(0, \pi/4)$  but not on any larger disk centered at 0. Since  $h(z) = 0$  at these two points but  $g(z)$  is nonzero there,  $f$  actually blows up at those points, so there is no function agreeing with  $f$  on  $D(0, \pi/4)$  but analytic on a larger disk. Thus, the radius of convergence of the power series must be  $R = \pi/4$ .

5. Let  $D = \{z \in \mathbb{C} : \operatorname{Re} z \leq -2\}$ . Prove that  $\sum_{n=1}^{\infty} n^z$  converges uniformly on  $D$ , where  $n^z$  denotes  $e^{z \operatorname{Log} n}$ .

**Proof.** For each integer  $n \geq 1$ , define  $M_n = 1/n^2$ . For any  $z \in D$  and any  $n \geq 1$ , writing  $z = x + iy$ , we have

$$|n^z| = |e^{(x+iy) \log n}| = |e^{x \log n} e^{iy \log n}| = |e^{x \log n}| = n^x \leq n^{-2} = M_n.$$

Meanwhile,  $\sum_{n \geq 1} M_n = \sum_{n \geq 1} n^{-2}$  converges by the  $p$ -test, since  $2 > 1$ . Thus, the original sum converges uniformly on  $D$ , by the Weierstrass M-Test. QED

6. Let  $E = \{z \in \mathbb{C} : |z| \geq 7\}$ . Prove that  $\sum_{k=1}^{\infty} \frac{z^k}{5k - z^{3k}}$  converges uniformly on  $E$ .

**Proof.** For each integer  $k \geq 1$ , define  $M_k = 1/7^k$ . For any  $z \in E$  and any  $k \geq 1$ , note that

$$|z|^{3k} - |z|^{2k} \geq |z|^{2k}(|z|^k - 1) \geq 49^k(7 - 1) \geq 5k,$$

and hence  $|5k - z^{3k}| \geq |z|^{3k} - |5k| \geq |z|^{2k}$ . Thus,  $\left| \frac{z^k}{5k - z^{3k}} \right| \leq \frac{|z|^k}{|z|^{2k}} = \frac{1}{|z|^k} \leq \frac{1}{7^k} = M_k$ . Meanwhile,  $\sum_{k \geq 1} M_k = \sum_{k \geq 1} 7^{-k}$  converges by the Geometric Series Test (since  $r = 1/7$  has  $|r| < 1$ ). Thus, the original sum converges uniformly on  $E$ , by the Weierstrass M-Test. QED

7. Let  $f$  be an entire function with the property that for all  $z \in \mathbb{C}$ ,

$$f(z+1) = f(z+i) = f(z).$$

Prove that  $f$  is constant.

**Proof.** Let  $E$  be the closed square with vertices at 0, 1,  $i$ ,  $1+i$ ; that is,  $E = \{x + iy : x, y \in [0, 1]\}$ . Since  $E$  is closed and bounded, it is compact. Since  $f$  is entire, the real-valued function  $|f(z)|$  is continuous on  $E$  and hence attains a maximum value  $M \in \mathbb{R}$ . We claim that  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$ . Given  $z \in \mathbb{C}$ , write  $z = x + iy$ , and write  $x = m + s$  and  $y = n + t$  with  $m, n \in \mathbb{Z}$  and  $s, t \in [0, 1)$ . By repeated application of the equations of the hypotheses [technically by induction on  $m, n \geq 0$  and then by a quick extra argument for  $m, n \leq -1$ ], we have  $f(z) = f(s + it)$ . Since  $s + it \in E$ , we have  $|f(z)| = |f(s + it)| \leq M$ , proving the claim.

Thus,  $f$  is both entire and bounded. By Liouville's Theorem,  $f$  is constant. QED