Math 345, Fall 2024 Professor Rob Benedetto

Midterm Exam 1, Take-Home

~~Due **Thursday, October 24** in Gradescope by **11:59 pm ET**~~

EXTENDED to **Friday, October 25**, in Gradescope by **11:59 pm ET**

(You are welcome to submit it before that, if you want.)

Instructions: **Answers must be written neatly and legibly, and tagged to the correct problem numbers on Gradescope.**

You must fully justify your answers. (So on computational problems, show all steps and explain your thinking; on proof problems, give rigorous proofs.)

You may use any theorems from class (through Section IV.1), from Sections I.1–IV.1 of the book, or from assigned homework problems, as well as standard theorems from single variable and multivariable calculus (Math 111, 121, 211).

However, you must clearly verify all the hypotheses of any theorem you use, and you must reference the source of such theorems. (E.g., "By the Theorem on page 50..., or "By Green's Theorem...") If you are not sure whether or not some argument or statement requires further justification, please ask me about it.

You may use Gamelin (Sections I.1–IV.1), your notes, your own old homework, and any materials from the **course** websites, including handouts, problem solutions, and videos from the course. You may also consult a textbook on single and/or multivariable calculus (like Stewart's *Calculus*), provided it does not cover material beyond standard multivariable calculus.

As noted above, you may quote the results of any assigned homework problems (whether or not you correctly solved those problems yourself), but **not** challenge problems.

You may NOT use other books, online information, AI, calculators, or any other outside sources.
You also may NOT discuss the problems with anyone other than me.

But you **should** feel free to talk to me about anything on the exam.

There are seven problems, totalling 100 points.

1. **(10 points)** Find all complex roots of $z^7 + 8iz = 0$.

2. **(12 points)** Sketch the region

$$D = \left\{ z \in \mathbb{C} : \frac{\pi}{6} < \operatorname{Arg} z < \frac{\pi}{3}, \text{ and } 1 < |z| < 2 \right\}.$$

Then compute and sketch $f(D)$, where $f(z) = iz^3 - 7$.

(*Hint*: it might be easier to find the image of D under z^3 first.)

3. **(14 points)** Let $u(x, y) = x^4 + y^4 + a(x^2y^2 - 2x + 4y) + e^{by} \sin(5x)$.
   (a) Find constants $a, b \in \mathbb{R}$ such that u is harmonic on $\mathbb{C}$.
   (b) For those choices of a, b, find a harmonic conjugate v for u on $\mathbb{C}$.
   (c) For the same choice of a, b, and v, express $f = u + iv$ in terms of $z(= x + iy)$ only.

(Problems continue on next page)

4. **(14 points)** Let γ_1 be the quarter-circle path (along $|z| = 2$) from 2 to $2i$, and let γ_2 be the straight-line segment from $2i$ to -2. Let $f(z) = 3z - (\overline{z})^2$. Compute

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

5. **(15 points)** Let $f(z) = \dfrac{z^3 e^{iz}}{z^5 + 10}$, and for any real number $R > 2$, let γ_R be the path from R to $-R$ along the upper half of the circle $|z| = R$. Use the ML-estimate to prove that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0.$$

6. **(15 points)**

6a. Fix $c \in \mathbb{C}$ with $0 < |c| < 1$. Let $f(z) = \dfrac{z - c}{\bar{c}z - 1}$.

Prove that f maps the open unit disk $D(0, 1)$ one-to-one and onto itself.

6b. For any $a, b \in D(0, 1)$, prove that there is a linear fractional transformation $g(z)$ such that g maps $D(0, 1)$ one-to-one and onto itself, with $g(a) = b$.

7. **(20 points)** Let $f : [3, 8] \to \mathbb{C}$ be a continuous function, and let $D = \mathbb{C} \setminus [3, 8]$. For all $z \in D$, define

$$g(z) = \int_3^8 \frac{f(t)}{t - z}\,dt.$$

Of course, g is defined on D because for any $z \in D$, the function $f(t)/(t - z)$ is a continuous function of $t \in [3, 8]$, and so the integral makes sense.

Prove that g is (complex) differentiable on D. (In fact, g is analytic on D, but I am only asking you to show it's differentiable.)

(*Hint*: Use the limit definition of the derivative. At some point, you will probably need the ML-estimate to show that the limit really converges to what it looks like it's converging to.)

OPTIONAL BONUS. **(2 points)**
Let $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ and } |z - 5i| > 3\}$, i.e., the open upper half-plane with the closed disk $\overline{D}(5i, 3)$ removed.
For any real number $0 < r < 1$, define U_r to be the annulus $U_r = \{z \in \mathbb{C} : r < |z| < 1\}$.
Find a real number $0 < r < 1$ and a function $f : D \to U_r$ that is analytic, one-to-one, and onto.