Math 280, Fall 2026

Solutions to Homework #9

1. Textbook Section 1.6.2, #5: For any integer $n \geq 2$, prove that the only graph G of order n for which $\chi(G) = n$ is the complete graph K_n. That is, prove that for any $n \geq 2$ and any graph G of order n, we have $\chi(G) = n$ if and only if G is complete.

Proof. ($\Rightarrow$): We prove the contrapositive. Assume G is *not* complete. Label the vertices of G as $x_1, \ldots, x_n$. Since G is not complete, we may assume without loss that x_{n-1} and x_n are not adjacent. Define $K : V(G) \to \{1, \ldots, n-1\}$ by

$$K(x_i) = \begin{cases} i & \text{if } 1 \leq i \leq n-1, \\ n-1 & \text{if } i = n. \end{cases}$$

We claim that K is a proper coloring. Indeed, for any edge $e \in E(G)$, the two vertices of e are $x_i \neq x_j$ with $i < j$.
If $j \leq n-1$, then $K(x_i) = i \neq j = K(x_j)$.
If $j = n$, then $i \neq n$ (since $i < j$) but also $i \neq n-1$ (since there x_{i-1} and x_i are not adjacent). Thus, $i \leq n-2$, so $K(x_i) = i \neq n-1 = K(x_j)$, proving our claim.
Thus, G is $(n-1)$-colorable, so $\chi(G) \leq n-1$. QED ($\Rightarrow$)

($\Leftarrow$): If G is complete, then $G = K_n$. From the book (and class) we have $\chi(K_n) = n$, as desired. QED

---

2. Textbook Section 1.6.2, #6: Prove that for any $n \geq 1$ and any graph G of order n, we have

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G).$$

Proof. For the first inequality, let $k = \chi(G)$, and let $K : V(G) \to \{1, \ldots, k\}$ be a proper k-coloring of G. Then for each color $i = 1, \ldots, k$, the set

$$S_i = K^{-1}(i) = \{v \in V(G) : K(v) = i\}$$

is an independent set of vertices, since no two vertices of the same color are adjacent. Therefore, $|S_i| \leq \alpha(G)$ for each $i = 1, \ldots, k$. Since $V(G)$ is the disjoint union $V(G) = S_1 \cup S_2 \cup \cdots \cup S_k$, we have

$$n = |V(G)| = |S_1| + |S_2| + \cdots + |S_k| \geq k\alpha(G) = \alpha(G)\chi(G).$$

Dividing both sides by $\alpha(G) > 0$, we have $\dfrac{n}{\alpha(G)} \leq \chi(G)$, as desired.

For the second inequality, let $m = \alpha(G)$, and let $S \subseteq V(G)$ be an independent set of m vertices. Write the complement of S as

$$V(G) \setminus S = \{x_1, x_2, \ldots, x_{n-m}\}$$

Let $\ell = n + 1 - \alpha(G) = n - m + 1$. Define an ℓ-coloring $K : V(G) \to \{1, 2, \ldots, \ell\}$ by

$$K(v) = \begin{cases} i & \text{if } v = x_i \text{ for some } 1 \leq i \leq n-m, \\ n-m+1 & \text{if } v \in S. \end{cases}$$

Then K is a proper coloring, because if vertices $u \neq w$ are adjacent, there are three cases:

Case 1: If $u, w \notin S$, then $u = x_i$ and $w = x_j$ for some $i \neq j$, so $K(u) = i \neq j = K(w)$.

Case 2: If $u \in S$ and $w \notin S$ (or vice versa), then $u = x_i$ and for some $1 \leq i \leq n-m$, so $K(u) = i \neq n-m+1 = K(w)$.

Case 3: If $u, w \in S$, then u and w are not adjacent by definition of independent set, so this case does not actually arise.

That is, K is a proper ℓ-coloring, and hence $\chi(G) \leq \ell = n - m + 1$, as desired. QED

3. Textbook Section 1.6.4, #4: Prove that for any tree T of order n, the chromatic polynomial of T is $c_T(k) = k(k-1)^{n-1}$.

Proof. We proceed by induction on $n \geq 1$.
For $n = 1$, the tree T is simply a single vertex, so $c_T(k) = k$, as desired.

For $n \geq 2$, assume the statement is true for trees of order $n-1$. Given a tree T of order n, we know that T has at least two leaves, so let $v \in V(T)$ be a leaf of T, and let $e = vw$ be the unique edge incident with v.
The graph T/e is the same as $T' = T - v$, since collapsing the edge e simply glues v to w. Thus, $c_{T/e} = c_{T'}$.
The graph $T - e$ has two components: the isolated vertex v and the graph T'. Thus, $c_{T-e}(k) = c_v(k)c_{T'}(k) = kc_{T'}(k)$.
But T' is a tree with $n-1$ vertices. By the inductive hypothesis, we have $c_{T'}(k) = k(k-1)^{n-2}$
Therefore, by Theorem 1.48, we have

$$c_T(k) = c_{T-e}(k) - c_{T/e}(k) = kc_{T'}(k) - c_{T'}(k) = (k-1)\big[k(k-1)^{n-2}\big] = k(k-1)^{n-1}$$

as desired. QED

---

4. Prove that for any $n \geq 3$, the chromatic polynomial of the cycle graph C_n is $c_{C_n}(k) = (k-1)\big((k-1)^{n-1} + (-1)^n\big)$.

Proof. We proceed by induction on $n \geq 3$. For $n = 3$, we have $C_3 = K_3$, so

$$c_{C_3}(k) = k(k-1)(k-2) = (k-1)(k^2-2k) = (k-1)\big((k-1)^2 - 1\big),$$

as desired.

For $n \geq 4$, assume the statement is true for trees of order $n-1$. Let e be any edge of C_n. Then $C_n - e$ is P_n, and C_n/e is a cycle graph on $n-1$ vertices, and hence is isomorphic to C_{n-1}.
Since P_n is a tree, we have

$$c_{C_n-e}(k) = c_{P_n}(k) = k(k-1)^{n-1}$$

by Problem 3. By the inductive hypothesis, we have

$$c_{C_n/e}(k) = c_{C_{n-1}}(k) = (k-1)\big((k-1)^{n-2} + (-1)^{n-1}\big).$$

Therefore, by Theorem 1.48, we have

$$\begin{aligned} c_{C_n}(k) &= c_{C_n-e}(k) - c_{C_n/e}(k) = k(k-1)^{n-1} - (k-1)\big((k-1)^{n-2} + (-1)^{n-1}\big) \\ &= (k-1)\Big(k(k-1)^{n-2} - (k-1)^{n-2} - (-1)^{n-1}\Big) = (k-1)\Big((k-1)(k-1)^{n-2} + (-1)^n\Big) \\ &= (k-1)\big((k-1)^{n-1} + (-1)^n\big). \end{aligned}$$

QED

---

5. Let $n \geq 2$, and let e be any edge of the complete graph K_n. Prove that K_n/e is isomorphic to K_{n-1}.

Solution/Proof. Label the vertices of K_n as $\{x_1, \ldots, x_n\}$. Relabeling if necessary, we may assume that the two vertices of e are x_{n-1} and x_n.
Thus, K_n/e has $n-1$ vertices $\{x_1, \ldots, x_{n-1}\}$, which are the same vertices as before except now with the old x_n merged into x_{n-1}.
In addition, for any $i < j \leq n-2$, there is an edge from x_i to x_j, because there was one before, and these two vertices were not disrupted.
Furthermore, for any $i < n-1$, there is an edge from x_i to x_{n-1}, because there was one before. (There was also one from x_i to x_n, and these two edges have now merged into one.)

Thus, for every $i \neq j$ between 1 and $n-1$, there is an edge from x_i to x_j. That is, K_n/e is a complete graph on $n-1$ vertices, i.e., $K_n/e \cong K_{n-1}$. QED

---

6. For any $n \geq 2$, let $H = K_n - e$ be the graph K_n with one edge removed. Prove that the chromatic polynomial of H is is $c_H(k) = k(k-1)\cdots(k-n+3)(k-n+2)^2$.

Proof. Theorem 1.48 applied to the graph K_n with the edge e yields

$$c_{K_n}(k) = c_H(k) - c_{K_n/e}(k).$$

We know that $c_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$. In addition, by Problem 5, we have $K_n/e \cong K_{n-1}$. Thus, rearranging the above equation gives

$$\begin{aligned} c_H(k) &= c_{K_n}(k) + c_{K_n/e}(k) = c_{K_n}(k) + c_{K_{n-1}}(k) \\ &= k(k-1)(k-2)\cdots(k-n+1) + k(k-1)(k-2)\cdots(k-n+2) \\ &= k(k-1)(k-2)\cdots(k-n+2)\big((k-n+1)+1\big) = k(k-1)\cdots(k-n+3)(k-n+2)^2, \end{aligned}$$

as desired. QED

---

7. Let $n \geq 1$ be a positive integer, and let $G = K_{2,n}$. Prove that the chromatic polynomial of G is $c_G(k) = k(k-1)\big((k-2)^n + (k-1)^{n-1}\big)$.

Proof. Let $a, b \in V(G)$ be the two vertices on the 2-side of $G = K_{2,n}$. Let Y be the set of the other n vertices.

For any integer $k \geq 1$, let D_k be the set of all proper colorings of G using (up to) k colors, so that each $f \in D_k$ is a function $f : V(G) \to \{1, \ldots, k\}$ such that no two adjacent vertices have the same value. Then define

$$D_{k,1} = \{f \in D \,|\, f(a) = f(b)\} \quad \text{and} \quad D_{k,2} = \{f \in D \,|\, f(a) \neq f(b)\},$$

so that $D_{k,1} \cap D_{k,2} = \varnothing$, with $D_k = D_{k,1} \cup D_{k,2}$. Thus, $|D_k| = |D_{1,k}| + |D_{2,k}|$.

Claim 1: $|D_{k,1}| = k(k-1)^n$.

Proof of Claim 1. To describe a coloring $f \in D_{k,1}$, there are k choices of the color $c \in \{1, \ldots, k\}$ such that $f(a) = f(b) = c$. The only edges run from a to each of the vertices $y \in Y$, and from b to each of the vertices $y \in Y$. Thus, for each $y \in Y$, there are $k-1$ choices of $f(y)$, since the only vertices adjacent to y are a, b, and hence $f(y)$ may be any of the k colors other than c.
Since $|Y| = n$, it follows that for each c, there are $(k-1)^n$ choices of $f \in D_{1,k}$ with $f(a) = f(b) = c$. Thus, $|D_{k,1}| = k(k-1)^n$, proving Claim 1.

Claim 2: $|D_{k,2}| = k(k-1)(k-2)^n$.

Proof of Claim 2. To describe a coloring $f \in D_{k,2}$, there are k choices of the color $c \in \{1, \ldots, k\}$ such that $f(a) = c$, and then $k-1$ remaining choices of the color d such that $f(b) = d$, since by definition of $D_{2,k}$, we have $f(d) \neq f(c)$.
The only edges run from each of a, b to each of the vertices $y \in Y$. Thus, for each $y \in Y$, there are $k-2$ choices of $f(y)$, since the only vertices adjacent to y are a, b, and hence $f(y)$ may be any of the $k-2$ colors other than c, d.
Since $|Y| = n$, it follows that for each choice of $c \neq d$, there are $(k-2)^n$ choices of $f \in D_{2,k}$ with $f(a) = c$ and $f(b) = d$.
Thus, $|D_{k,2}| = k(k-1)(k-2)^n$, proving Claim 2.

Summing, we have $c_G(k) = |D_{1,k}| + |D_{2,k}| = k(k-1)^n + k(k-1)(k-2)^n = k(k-1)\big((k-2)^n + (k-1)^{n-1}\big)$.