Math 280, Fall 2026

Solutions to Homework #8

1. Textbook Section 1.5.2, #1:
Let G be a connected planar graph of order 24, and suppose that G is regular of degree 3. How many regions are there in a planar representation of G?

Solution. By an old theorem, the number q of edges satisfies

$$2q = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} 3 = 24 \cdot 3 = 72,$$

so $q = 36$. Letting r be the desired number of regions, Euler tells us

$$2 = n - q + r = 24 - 36 + r = r - 12.$$

Thus, $r = 14$, i.e., there are 14 regions

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2. Textbook Section 1.5.2, #4: Let G be a connected planar graph of order $n \geq 3$, and suppose that G is K_3-free. (That is, G has no cycles of length 3.) Prove that the number q of edges of G satisfies $q \leq 2n - 4$.

Proof. Fix a planar representation of G, with r regions.

Case 1: $r = 1$. Then Euler tells us $n - q + 1 = 2$, i.e., $q = n - 1$. Since $n - 3 \geq 0$, it follows that $q \leq (n-1) + (n-3) = 2n - 4$, as desired.

Case 2: $r \geq 2$. Then by a proposition from class (stated informally in the book), each region R has $b(R) \geq 3$; but $b(R) = 3$ can only be attained if those three edges form a 3-cycle, which our hypotheses forbid. So every region R has $b(R) \geq 4$. Since each edge bounds at most two regions, we have

$$4r = \sum_{R \text{ region}} 4 \leq \sum_{R \text{ region}} b(R) \leq \sum_{e \in E(G)} 2 = 2q,$$

and hence $2r \leq q$.
Euler tells us $n - q + r = 2$; multiplying by 2 and rearranging gives $4 + 2q - 2n = 2r$. Thus,

$$4 + 2q - 2n = 2r \leq q,$$

so $q \leq 2n - 4$, as desired. QED

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3. Textbook Section 1.5.2, #5:
Let G be a bipartite planar graph. Prove that $\delta(G) \leq 3$.

Proof. **Case 1**: G is connected. Let $n = |V(G)|$ and $q = |E(G)|$.
Since G is bipartite, it has no cycles of odd length, and in particular it is K_3-free. Therefore, by the previous problem, we have $q \leq 2n - 4$, and hence $4n \geq 2q + 4$.
Suppose $\delta(G) \geq 4$. Then for every $v \in V(G)$, we have $\deg(v) \geq 4$. Therefore, by an old theorem and the previous paragraph, we have

$$2q = \sum_{v \in V(G)} \deg(v) \geq 4n \geq 2q + 4.$$

Subtracting $2q$, we have $0 \geq 4$, a contradiction. Thus, $\delta(G) \leq 3$.

Case 2: G is disconnected. Let H be any connected component of G. Then by Case 1, we have $\delta(H) \leq 3$. That is, there is some $w \in V(H)$ with $\deg(w) \leq 3$. Since $w \in V(G)$, we have $\delta(G) \leq \deg(w) \leq 3$. QED

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4. Textbook Section 1.5.2, #7: Let G be of order $n \geq 11$. Prove that at least one of G or $\overline{G}$ is nonplanar.

Proof. Suppose not, i.e., both G and $\overline{G}$ are planar. Let $n = |V(G)| = |V(\overline{G})|$, let $q_1 = |E(G)|$, and let $q_2 = |E(\overline{G})|$.
By Theorem 1.33, we have $q_1 \leq 3n - 6$ and $q_2 \leq 3n - 6$.
On the other hand, $q_1 + q_2 = n(n-1)/2$ is the total number of possible edges on n vertices, so we have

$$n^2 - n = 2q_1 + 2q_2 \leq 2(3n-6) + 2(3n-6) = 12n - 24,$$

and hence

$$n^2 - 13n + 24 \leq 0.$$

However,

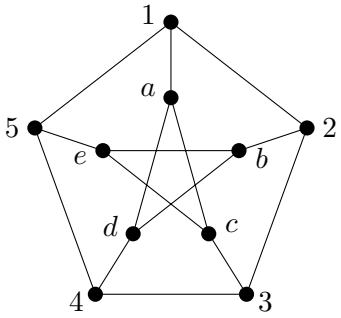
$$n^2 - 13n + 24 = (n^2 - 13n + 22) + 2 = (n-2)(n-11) + 2 \geq 2,$$

since $n \geq 11$. Thus, $2 \leq 0$, a contradiction. Thus, at least one of G or $\overline{G}$ must be nonplanar. QED

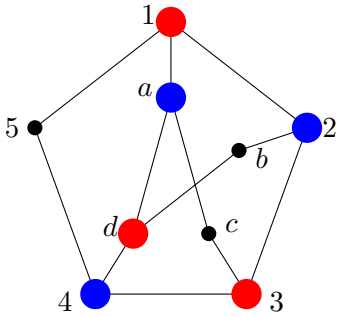
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5. Textbook Section 1.5.4, #2: Use Kuratowski's Theorem to prove that the Petersen graph G is nonplanar. More specifically, show that G has a subgraph that is a subdivision of $K_{3,3}$

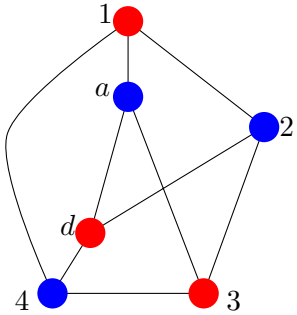
Solution/Proof. Here's the Petersen graph G:



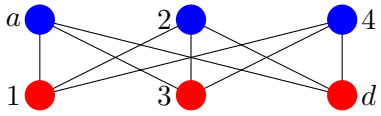
Consider the following subgraph $G' = G - e$, formed by deleting vertex e:



I've highlighted three vertices in red and three in blue. The above graph is a subdivision of this one, which I'll call H:



by adding the vertex b in the middle of the edge $2d$, adding the vertex c in the middle of the edge $a3$, and adding the vertex 5 in the middle of the edge 14. This last graph H can be redrawn as:



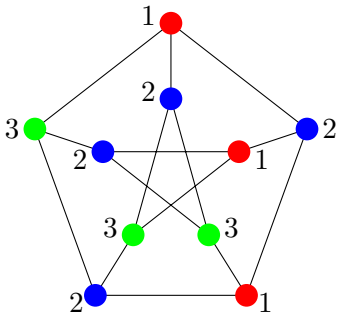
which is $K_{3,3}$. Thus, the subgraph G' of G is a subdivision of $H \cong K_{3,3}$, so by Kuratowski's Theorem, G is not planar. QED

Note: There are many other ways to exhibit a subdivision of $K_{3,3}$ as a subgraph of the Petersen graph.

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6. Textbook Section 1.6.1, #1(d): Determine the chromatic number of the Petersen graph. As always, don't forget to justify your answer.

Solution/Proof. We claim the Petersen graph G has chromatic number $\boxed{\chi(G) = 3}$
To see this, first of all, here is a proper 3-coloring of G, obtained by applying the greedy algorithm to the ordering $1, 2, 3, 4, 5, a, b, c, d, e$ of the vertices as labeled previously, but now labeled with red=1, blue=2, green=3:

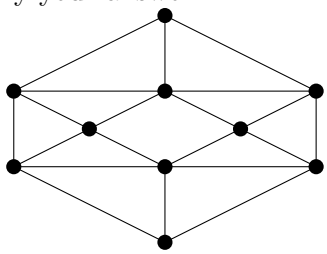


Thus, $\chi(G) \leq 3$.
On the other hand, G has a copy of C_5 as a subgraph (for example, the outer pentagon), so if there were a proper 2-coloring of G, then there would be a proper 2-coloring of the subgraph C_5. Since $\chi(C_5) = 3$, no such 2-coloring exists. Thus, $\chi(G) \geq 3$, as desired. QED

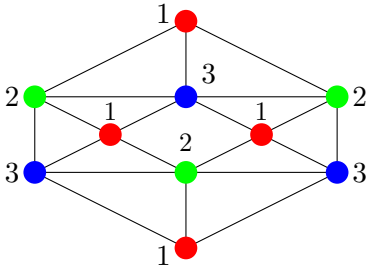
Note: There are many other proper 3-colorings of the Petersen graph. However, they all essentially look like the one above after some rotating or flipping of the graph, and perhaps permuting the three colors. (If you really want to be technical about it, you can get a non-isomorphic coloring by swapping the red and green vertices on the outer pentagon at the top and upper left. But then every proper 3-coloring of the Petersen graph is isomorphic either to the picture above or to the swapped version.)

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7. Textbook Section 1.6.1, #1(e): Determine the chromatic number of the Birkhoff Diamond, shown below. As always, don't forget to justify your answer.



Solution/Proof. We claim the Birkhoff Diamond B has chromatic number $\boxed{\chi(B) = 3}$
To see this, first of all, here is a proper 3-coloring of G, obtained by applying the greedy algorithm from ordering the vertices top to bottom and left to right, and labeled red=1, blue=2, green=3:



Thus, $\chi(B) \leq 3$.

On the other hand, B has a copy of C_3 as a subgraph (for example, the right triangle at the upper left), so if there were a proper 2-coloring of G, then there would be a proper 2-coloring of the subgraph C_3. Since $\chi(C_3) = 3$, no such 2-coloring exists. Thus, $\chi(B) \geq 3$, as desired. QED

Note: There are *not* really any other proper 3-colorings of the Birkhoff Diamond. The only change you can make is to shuffle the colors. Can you see how to prove that for *any* proper 3-coloring of this graph, the vertices colored red above must all be the same color, the vertices colored blue must all be the same color (different from the first), and the remaining three vertices must also all be the same color (different from the first two)?

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8. Textbook Section 1.6.1, #4 (slightly expanded): Let G be a graph, let $e \in E(G)$, and let $G' = G - e$. Prove that $\chi(G') \leq \chi(G) \leq \chi(G') + 1$.

Proof. Let $k = \chi(G')$ and $\ell = \chi(G)$.

First inequality: Since $\chi(G) = \ell$, there is a proper ℓ-coloring $K : V(G) \to \{1, \ldots, \ell\}$ of G. Since $V(G') = V(G)$ and $E(G') \subseteq E(G)$, it follows that K is also a proper ℓ-coloring of G'. Thus, $\chi(G') \leq \ell = \chi(G)$, as desired.

Second inequality: Since $\chi(G') = k$, there is a proper k-coloring $K' : V(G') \to \{1, \ldots, k\}$ of G'. Since $V(G') = V(G)$, the function K' is also a k-coloring but may not be proper, because it's possible that $K'(v) = K'(w)$, where $v, w \in V(G)$ are the two vertices of the edge e.
Define $L : V(G) \to \{1, \ldots, k, k+1\}$ by

$$L(x) = \begin{cases} K'(x) & \text{if } x \neq w, \\ k+1 & \text{if } x = w. \end{cases}$$

Then L is a $(k+1)$-coloring of G, and we claim it is proper. Indeed, for any adjacent $x, y \in V(G)$, there are two possibilities.

Case 1: Neither of x, y is w. Then xy is not the edge e, so x and y are adjacent in G'. Thus, $L(x) = K'(x) \neq K'(y) = L(y)$, as desired.

Case 2: One of x, y is w; without loss $x = w$. Then $y \neq w$, since they are adjacent, so $L(y) = K'(y) \in \{1, \ldots, k\}$. Thus, $L(x) = k + 1 \neq L(y)$, as desired.

Having proven our claim that L is a proper $(k+1)$-coloring of G, it follows that $\chi(G) \leq k+1 = \chi(G')+1$. QED