Math 280, Fall 2026

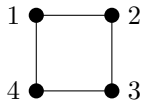
Solutions to Homework #7

1. For the graph $G = C_4$, determine:

   (a) is it Eulerian? (b) is it Hamiltonian?

   (c) is it traceable? (d) what is its independence number $\alpha(G)$?

As always, be sure to (briefly) justify your answers.



Solution. For reference, here's the graph:
(a): Yes, C_4 is Eulerian because the walk $1, 2, 3, 4, 1$ is a circuit and uses all four edges. [Or, alternatively, because the graph is connected and every vertex has degree 2, which is even.]

(b): Yes, C_4 is Hamiltonian because the (same) walk $1, 2, 3, 4, 1$ is a cycle and uses all four vertices.

(c): Yes, C_4 is traceable because it is Hamiltonian. (Or specifically, the slightly shorter walk $1, 2, 3, 4$ is a path and uses all four vertices.)

(d): The independence number is $\alpha(C_4) = 2$ because the vertices $1, 3$ are not adjacent, so $\{1, 3\}$ is an independent set. Well, that only shows $\alpha(C_4) \geq 2$. But no larger set is independent, because if S is a set of at least three vertices of C_4, then there are $\binom{3}{2} = 3$ possible edges between them, but only two pairs of vertices (namely $\{1, 3\}$ and $\{2, 4\}$) that do not form edges of C_4, so S must contain at least two adjacent vertices, proving that $\alpha(C_4) < 3$, and hence it is indeed exactly 2.

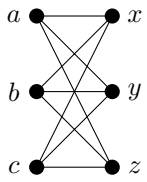
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2. For the graph $G = K_{3,3}$, determine:

   (a) is it Eulerian? (b) is it Hamiltonian?

   (c) is it traceable? (d) what is its independence number $\alpha(G)$?

As always, be sure to (briefly) justify your answers.



Solution. For reference, here's the graph:
(a): No, $K_{3,3}$ is not Eulerian by Theorem 1.20, because $\deg(a) = 3$ is odd. [In fact, all six vertices have odd degree, but we only need one.]

(b): Yes, $K_{3,3}$ is Hamiltonian because the walk a, x, b, y, c, z, a is a cycle and uses all six vertices.

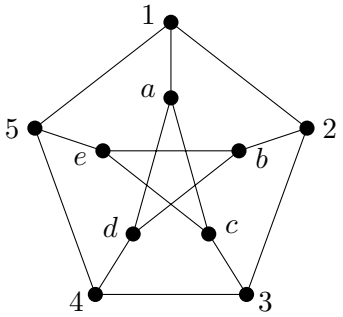
(c): Yes, $K_{3,3}$ is traceable because it is Hamiltonian. (Or specifically, the slightly shorter walk a, x, b, y, c, z is a path and uses all six vertices.)

(d): The independence number is $\alpha(K_{3,3}) = 3$ because the vertices a, b, c are pairwise non-adjacent, so $\{a, b, c\}$ is an independent set. And any set S of four or more vertices must contain at least one of a, b, c and at least one of x, y, z. But then those two vertices are adjacent to one another, so S cannot be independent; that is, $\alpha(K_{3,3}) < 4$, and hence $\alpha(K_{3,3}) = 3$.

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3. Textbook Section 1.4.3, problem 4: Find the connectivity and the independence number of the Petersen graph. Make sure to prove your answers!

Solution/Proof. Let G denote the Petersen graph:



We claim the connectivity is $\boxed{\kappa(G) = 3}$

Certainly removing the vertices $5, a, 2$ (i.e., the neighbors of vertex 1) leaves 1 as an isolated vertex, so $\{5, a, 2\}$ is a cut set, and hence $\kappa(G) \leq 3$.

On the other hand, for any set S of two or fewer vertices, there are three possibilities:

Case 1: S is contained in the outer cycle $\{1, 2, 3, 4, 5\}$. Then $G - S$ still has the inner cycle $\{a, b, c, d, e\}$ with all five edges among them, along with the edges from each of the remaining outer vertices to the inner vertices $\{a, b, c, d, e\}$. Thus, $G - S$ is connected.

Case 2: S is contained in the inner cycle $\{a, b, c, d, e\}$. This case is similar to Case 1, with the roles of the inner and outer cycles reversed.

Case 3: S has one vertex x in the outer cycle, and a second vertex y in the inner cycle. Then the remaining four vertices of the outer cycle are connected to one another via the remaining outer edges. Similarly, the remaining four vertices of the inner cycle are connected to one another. And at least one of the edges $a1$, $b2$, $c3$, $d4$, or $e5$ is still an edge of $G - S$ (in fact, at least three of them are, since at most two were removed), so the inner vertices are connected to the outer ones, and hence $G - S$ is connected.

Thus, G has no cut set with 2 or fewer vertices, proving our claim that $\kappa(G) = 3$. QED

We claim the independence number is $\boxed{\alpha(G) = 4}$

Certainly $U = \{1, 3, d, e\}$ is an independent set, since no two of these four vertices are adjacent. So $\alpha(G) \geq 4$. On the other hand, for any set S of five or more vertices, there are two possibilities:

Case 1: S includes at least three of the outer vertices $\{1, 2, 3, 4, 5\}$. Without loss, one of them is 1, which means that if S were to be independent, we must have $2, 5 \notin S$. But then by process of elimination, we have $3, 4 \in S$, which are adjacent. Thus, S cannot be an independent set.

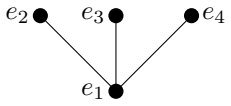
Case 2: S includes at most two of the outer vertices; then since $|S| \geq 5$, S must include at least three of the inner vertices $\{a, b, c, d, e\}$. This case is now similar to Case 1, with the roles of the inner and outer cycles reversed.

Thus, G has no independent set with 5 or more vertices, proving our claim that $\alpha(G) = 4$. QED

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4. Textbook Section 1.4.3, problem 8: Let G be a graph. Prove that the line graph $L(G)$ is claw-free.

Proof. Given any four-element subset $S = \{e_1, e_2, e_3, e_4\}$ of $V(L(G)) = E(G)$, suppose that the induced subgraph $\langle S \rangle$ of $L(G)$ is a claw, and without loss that e_1 is the vertex the others are joined to. That is, the induced subgraph $\langle S \rangle$ in $L(G)$ is:



Back in the graph G, let $v, w \in V(G)$ be the two vertices of e_1. Since in $L(G)$, we have that $e_1, e_2 \in L(G)$ are adjacent vertices, this means that in G, the edge e_2 is incident with either v or w, by the definition of $L(G)$.

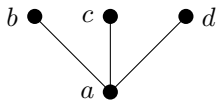
Similarly, the edges e_3 and e_4 of G are each incident with either v or w. Thus, still in G, at least two of e_2, e_3, e_4 must both be incident with one of v or w. Renaming if necessary, we may assume without loss that e_2 and e_3 are both incident with v.

But then by definition of $L(G)$, we have that the vertices $e_2, e_3 \in L(G)$ are joined by an edge of $L(G)$. But then the induced graph $\langle S \rangle$ above would have an edge between e_2 and e_3, contradicting the claw graph above. By this contradiction, $\langle S \rangle$ is *not* a claw. QED

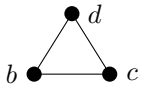
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5. Textbook Section 1.4.3, problem 9: Let G be a K_3-free graph. Prove that its complement, $\overline{G}$ is claw-free.

Proof. We will prove the contrapositive: if $\overline{G}$ has an induced subgraph that is a claw, then G has an induced subgraph that is K_3.
So we are assuming that there is a four-element subset $S = \{a, b, c, d\}$ of $V(\overline{G}) = V(G)$ such that the induced subgraph $\langle S \rangle$ is a claw, and without loss, we may assume that a is the vertex the others are joined to, like this:



But then none of the potential edges bc or bd or cd is an edge of $\overline{G}$, because if it were, it would have been part of the induced subgraph $\langle S \rangle$. By definition of $\overline{G}$, then, all three of the edges bc and bd and cd must be edges of G.
Thus, if we define $T = \{b, c, d\}$, then the induced subgraph $\langle T \rangle$ of G includes those three edges, i.e., it is

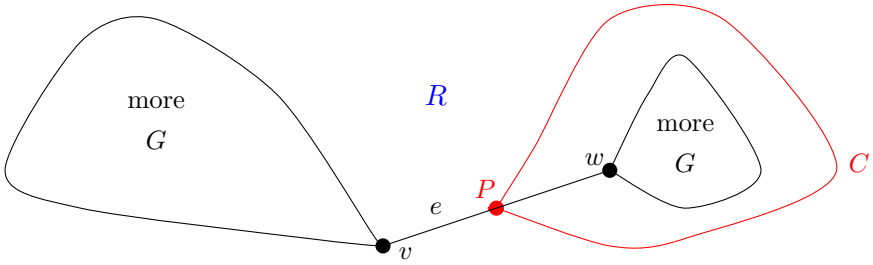


which is a copy of $K_3 = C_3$. Thus, G has an induced subgraph isomorphic to K_3. QED

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6. Textbook Section 1.5.1, converse of problem 6: Let G be a planar graph, and let $e \in E(G)$. Suppose that in some planar representation of G, the edge e does *not* bound a region. Prove that e is a bridge.

Proof. Consider a planar representation of G, so that the edge e shows up as a curve in the plane. [Remember from Math 211 that a straight line segment is a kind of curve.] Our hypothesis says that both sides of e abut the same region R. Let v and w be the two vertices of e.
Pick a point P in the middle of the curve e somewhere. Since R is a single connected region in the plane, there is another curve C (which is *not* part of the graph) that starts at P, runs through R without crossing itself, and ends back at P, but from the other side, like this:

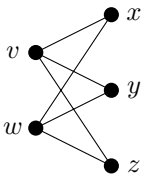


[The graph is in black, with probably lots of other edges and vertices in the portions labelled "more G", and the curve C — which, again, is *not* part of the graph, and which passes through the region R — is in red.]
Note that C is a simple closed curve in the plane — it starts and ends at the same point P, and it doesn't meet itself except at that start and end. So one of the two vertices of e is inside C — without loss, w is inside C, as in the figure above — and the other vertex is outside. Also, because C stays in R, it does not touch the graph G except at its start/end point P. So there cannot be a path in $G - e$ from w to v, because the graph $G - e$ does not cross the curve C anywhere at all. So $G - e$ is a disconnected graph, and hence e is a bridge. QED

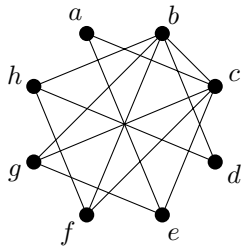
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7. Textbook Section 1.5.1, problem 1: Find planar representations of each of the following graphs:

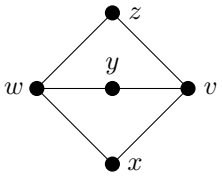
(a):



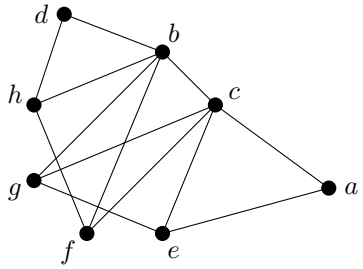
(b):



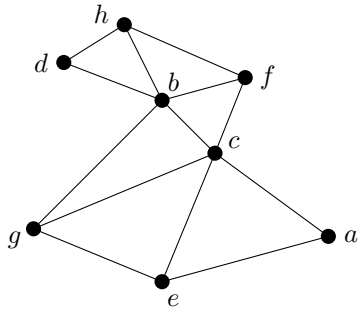
Solution. (a): [Inspired by imagining flipping vertex v to the other side of the vertical line with x, y, z.] Here is a planar representation of graph (a):



(b): Rather than just writing down an answer, here's a way of discovering it. First, inspired by the fact that the vertices a and d are of degree 2, flip them each to the other side of the two vertices they're joined to. That gives:



Now, to uncross the remaining lines, flip both vertices f and h to the other side of the path d, b, c, a, and we have our desired planar representation:



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8. Textbook Section 1.5.1, problem 8, slightly modified: Prove that there exist planar graphs G_1 and G_2 that have the same number n of vertices, the same number q of edges, and the same number r of regions, **but** which are not isomorphic.

Solution/Proof. Here is one way to produce two such graphs:



Clearly, both graphs are planar. Moreoverl, both have $n = 4$ vertices, $q = 4$ edges, and $r = 2$ regions: the region inside the square or triangle, and the region outside.
However, they are *not* isomorphic, because the vertex v in graph G_2 has $\deg(v) = 1$, whereas every vertex of G_1 has degree 2.