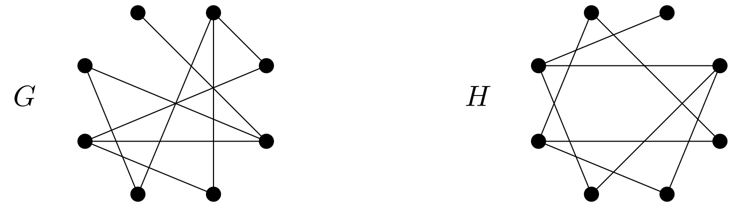
Math 280, Fall 2026

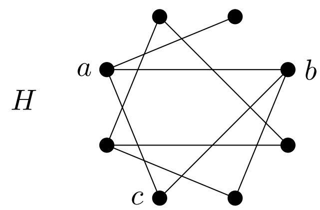
Solutions to Homework #3

1. Textbook, Section 1.1.3, Problem 9. Consider the following two graphs:



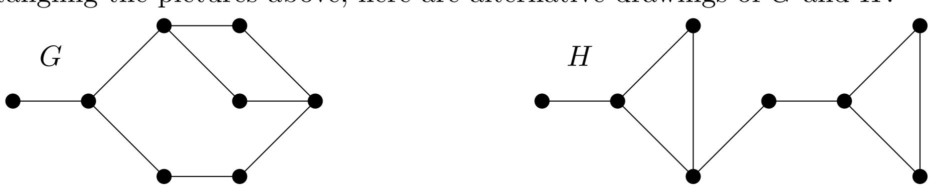
Verify that G and H have the same order, same size, and same degree sequence. Then prove that in spite of that, G and H are *not* isomorphic.

Answer/Proof: By inspection, the orders are $|V(G)| = 8 = |V(H)|$, the sizes are $|E(G)| = 9 = |E(H)|$, and the degree sequences are both 3, 3, 3, 2, 2, 2, 2, 1
However, if we label some of the vertices of H as follows:



we see that the vertices a, b, c form a cycle of length 3, involving two of the vertices of degree 3.
However, there is no cycle of length 3 in graph G, and there certainly isn't one involving two of the vertices of degree 3. Indeed, only two of the degree-3 vertices have an edge between them at all (the one horizontal-running edge in the picture of G), and these two vertices have no adjacent vertices in common.
An isomorphism would take the cycle a, b, c, a of length 3 to another cycle x, y, z, x of length 3 (and involving two vertices of degree 3), so since G doesn't have such a cycle, there cannot be an isomorphism.
QED

Side note: There are other ways to see G and H are not isomorphic. For example, H has a second cycle of length three that could be used instead. Or, rearranging the pictures by starting at the degree 1 vertex and untangling the pictures above, here are alternative drawings of G and H:



This new picture makes it clear that H has three bridges (single edges that can be removed to disconnect the graph), whereas G only has the one (from the vertex of degree 1).

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2. Suppose G is a graph that has 10 edges and 6 vertices, and suppose that the degrees of five of those vertices are $2, 2, 3, 4, 4$, and the sixth has some degree n.
(a) Find the integer n, i.e., the degree of the sixth vertex.
(b) Is G connected? (Yes, no, or maybe?) If "yes" or "no", prove it; if "maybe", draw two examples of such a graph G: one that is connected and one that is not.

Solution/Proof. (a): By an early theorem, the sum of the degrees of all the vertices is twice the number of edges, and hence

$$2 + 2 + 3 + 4 + 4 + n = 2 \cdot 10, \qquad \text{i.e.,} \qquad n = 5.$$

(b): Yes, G is connected. By part (a), there is a vertex v with $\deg(v) = 5$, and hence v must be adjacent to each one of the other five vertices. Therefore, the connected component of G that contains v also contains all six vertices.

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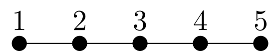
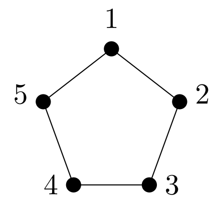
3. For each of the graphs P_5, C_5, and K_5:
(a) draw the graph
(b) find the eccentricity of each vertex
(c) find the radius and diameter of the graph
(d) find its adjacency matrix.

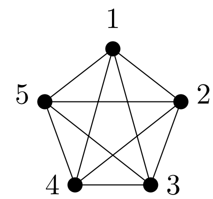
Proof. (a): Here are the three graphs:



P_5



C_5



K_5

(b,c): For P_5, by inspection (e.g. the shortest path from vertex 1 to vertex 5 has 4 edges), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(5) = 4, \qquad \mathrm{ecc}(2) = \mathrm{ecc}(4) = 3, \qquad \mathrm{ecc}(3) = 2,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 4.$$

For C_5, by inspection (e.g. the shortest path from vertex 1 to vertex 3 or 4 has 2 edges), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(4) = \mathrm{ecc}(5) = 2,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 2.$$

For K_5, by inspection (the shortest path between any two distinct vertices is a single edge), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(4) = \mathrm{ecc}(5) = 1,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 1.$$

(d): By inspection, the adjacency matrices are:

$$P_5 : \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_5 : \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad K_5 : \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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4. Textbook, Section 1.2.1, Problem 5:
Let G be a graph, and let $u, v \in V(G)$ be adjacent vertices. Prove that their eccentricities $\mathrm{ecc}(u)$ and $\mathrm{ecc}(v)$ differ by at most 1.

Proof. The shortest path from u to v is the length 1 path u, v, because of the edge joining them. (Note that they must be distinct, again because of the edge joining them, so their distance apart is not zero.) That is, $d(u, v) = d(v, u) = 1$.

Let $m = \mathrm{ecc}(u)$, so that there is some vertex $a \in V(G)$ such that $d(u,a) = m$. By the triangle inequality [from Optional Video 9], then we have

$$\mathrm{ecc}(u) = m = d(u,a) \le d(u,v) + d(v,a) = 1 + d(v,a).$$

Hence, by definition of the eccentricity of v,

$$\mathrm{ecc}(v) \ge d(v,a) \ge m - 1 = \mathrm{ecc}(u) - 1.$$

Reversing the roles of u and v, the same argument also yields

$$\mathrm{ecc}(u) \ge \mathrm{ecc}(v) - 1, \quad \text{i.e.,} \quad \mathrm{ecc}(v) \le \mathrm{ecc}(u) + 1.$$

Combining these two inequalities gives $\mathrm{ecc}(u) - 1 \le \mathrm{ecc}(v) \le \mathrm{ecc}(u) + 1$, as desired. QED

Note: If you don't want to quote the triangle inequality (from an optional video), here is a more direct way to see that $d(u,a) \le 1 + d(v,a)$:
Let P be a path from v to a be a path of shortest length, and write P as $v = x_1, x_2, \ldots, x_k = a$, which has length $k-1$, so that $k - 1 = d(v,a)$.
Then define a walk W from u to a by $u, v = x_1, x_2, \ldots, x_k = a$, i.e., by adding the edge uv to the beginning of the path P. The walk W has length $k = d(v,a) + 1$, but it might not be a path.
By an old theorem, W contains a path Q from u to a, so that the length of Q must be at most the length k of W. But of course, Q might not be a *shortest* path from u to a, but certainly the shortest such path is at most the length of Q, and hence $d(u,a) \le k = d(v,a) + 1$, as desired.

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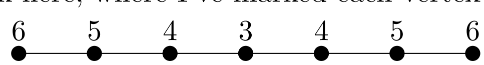
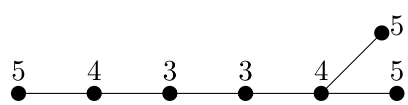
5. Textbook, Section 1.2.1, Problem 8(a,b,c):
(a) Draw a graph of order 7 that has radius 3 and diameter 6.
(b) Draw a graph of order 7 that has radius 3 and diameter 5.
(c) Draw a graph of order 7 that has radius 3 and diameter 4.

Solution. (a): P_7 does the trick here, where I've marked each vertex with its eccentricity:



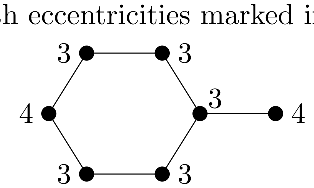
Noting the min and max eccentricities shows $\mathrm{rad}(P_7) = 3$ and $\mathrm{diam}(P_7) = 6$, as desired.

(b): Let G be the following graph, with eccentricities marked in:



Noting the min and max eccentricities shows $\mathrm{rad}(G) = 3$ and $\mathrm{diam}(G) = 5$, as desired.

(c): Let H be the following graph, with eccentricities marked in:



Noting the min and max eccentricities shows $\mathrm{rad}(H) = 3$ and $\mathrm{diam}(H) = 4$, as desired.

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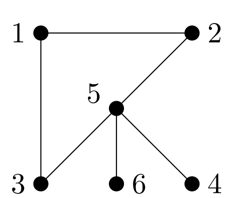
6. Let G be the following graph:



(a) Find the adjacency matrix A of G.
(b) Find all the walks of length 3 from vertex 1 to vertex 4. What is the total number of such walks, and (without computing A^3) what does this say about the matrix A^3?
(c) How many closed walks of length 3 are there in G? Without computing A^3, how would this number be related to the matrix A^3?
(d) Find the eccentricities of all the vertices of G.

Solution. (a): The adjacency matrix is $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(b): Any walk from 1 to 4 has to start by going from 1 to either 2 or 3, and end by going from 5 to 4. To get length 3, only one extra edge is allowed, so there are 2 such walks:

$$1, 2, 5, 4 \qquad \text{and} \qquad 1, 3, 5, 4.$$

Thus, both the $(1, 4)$ and $(4, 1)$ entries of A^3 must be 2, the number of walks of length 3 from vertex 1 to vertex 4, or from vertex 4 to vertex 1.
[I would also accept the weaker statement that just the $(1, 4)$ entry is 2, or just the $(4, 1)$ entry.]

(c): There are no closed walks of length 3 in G, because as no two consecutive vertices can coincide, such a walk would be of the form a, b, c, a with a, b, c being three distinct vertices. This excludes vertices 4 and 6, and the remaining four vertices form a 4-cycle but no shorter closed walks.
Thus, the number of closed walks of length 3 is 0. This means that there are 0 walks of length 3 from vertex i to vertex i, for each of $i = 1, \ldots 6$. That is, every entry on the diagonal of A^3 is 0.
[I would also accept the weaker statement that the trace of A^3 is 0.]

(d) The furthest vertex from either vertex 4 or 6 is vertex 1, which is distance 3 away. The vertices 2,3,5 are all distance at most 2 from any other vertices. Thus, the eccentricities are:

$$\mathrm{ecc}(1) = \mathrm{ecc}(4) = \mathrm{ecc}(6) = 3, \qquad \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(5) = 2.$$