Math 280, Fall 2026

**Solutions to Homework #1**

1. (8 points) Textbook, Section 1.1.2, Problem 1: If $G$ is a graph of order $n$, what is the maximum possible size of $G$?

**Solution**. Let $\mathcal{E}$ be the set of *all* two-element subsets of $V(G)$. Since there are $n$ vertices in $G$, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ such two-element subsets of $V(G)$.

Since $E(G) \subseteq \mathcal{E}$, we have $|E(G)| \le |\mathcal{E}| = \frac{n(n-1)}{2}$.

On the other hand, we can achieve this size by choosing $E(G) = \mathcal{E}$ to be the full set $\mathcal{E}$ of all such edges. Thus, the maximum possible size is $\binom{n}{2} = \frac{n(n-1)}{2}$.

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2. (12 points) Textbook, Section 1.1.2, Problem 2: Let $G$ be a graph of order $n \ge 2$. Prove that the degree sequence of $G$ has at least one pair of repeated entries.

**Proof**. For any vertex $v \in V(G)$, there are only $n-1$ other vertices (i.e., elements of $V(G) \smallsetminus \{v\}$), so there can be at most $n-1$ edges incident with $v$. Thus,

$$0 \le \deg(v) \le n-1 \quad \text{for all } v \in V(G).$$

Suppose, towards a contradiction, that all $n$ vertices have different degrees. Since there are $n$ vertices and $n$ integers between 0 and $n-1$, by the pigeonhole principle, there must be one vertex of each degree from 0 to $n-1$.

In particular, there must be vertices $a, b \in V(G)$ such that $\deg(a) = 0$ and $\deg(b) = n-1$. Since $n \ge 2$, these degrees are different, so $a \ne b$. Because $\deg(b) = n-1$, and there are only $n-1$ vertices in $V(G) \smallsetminus \{b\}$, there must be an edge from $b$ to every other vertex. In particular, there must be an edge from $b$ to $a$. But then there is at least one edge incident to $a$, so $0 = \deg(a) \ge 1$, which is a contradiction.

Thus, our supposition must be false. That is, there are at least two vertices with the same degree. Hence, there is at least one pair of repeated entries in the degree sequence of $G$. QED

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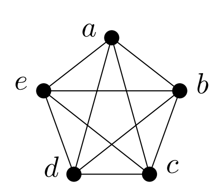
3. (6 points) Textbook, Section 1.1.2, Problem 3(c):
For this graph:




What is the maximum length of a circuit in this graph? Give an example of such a circuit.

**Solution**. There are 10 edges in this graph, so certainly no trail, and hence no circuit, can have length more than 10. And there are circuits of length 10, such as:

$$a, b, c, d, e, a, c, e, b, d, a$$

(i.e., trace the outer pentagon all the way around, and then the star all the way around). So this circuit has maximum length, because it uses all the edges, so there cannot be a longer one.

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4. (7 points) Let $G$ be a graph of odd order. Suppose that all the vertices of $G$ have the same degree $r$. Prove that $r$ is an even number.

**Proof**. Suppose (towards contradiction) that $r$ is odd. Let $n = |V(G)|$ be the number of vertices, which is odd, and let $m = |E(G)|$ be the number of edges. Then by the degree theorem, we have

$$nr = \sum_{i=1}^{n} r = \sum_{v \in V(G)} \deg(v) = 2|E(G)| = 2m.$$

But $2m$ is even, while $nr$ is odd, because $n$ and $r$ are both odd. This is a contradiction, so our supposition must have been wrong. That is, $r$ is even. QED

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5. (15 points) Textbook, Section 1.1.2, Problem 6: Prove that every closed walk of odd length in a graph contains a cycle of odd length.

**Proof**. Call our graph $G$. We prove this result for closed walks of length $2k+1$ by (strong) induction on $k \geq 0$.

**Base case**: For $k = 0$, let $W$ be a closed walk of length 1. We may write $W$ as $w_0, w_1$ with $w_0 = w_1$ (because $W$ is closed), but this contradicts the definition of walk, since $w_0w_1$ must be an edge, and in particular $w_0 \neq w_1$. That is, there are *no* closed walks of length 1, so the desired statement is vacuously true for the case $k = 0$.

**Inductive step**: For any $m \geq 2$, assume the statement is true for all $0 \leq k \leq m-1$; we will prove it for $m$. Let $W$ be a closed walk of length $2m+1$. Write $W$ as

$$w_0, w_1, \ldots, w_{2m+1}$$

with $w_0 = w_{2m+1}$. If all of $w_0, \ldots, w_{2m}$ are distinct, then $W$ is already a cycle of odd length, and we are done.
Otherwise, there are integers $i, j$ with $0 \leq i < j \leq 2m$ such that $w_i = w_j$. Let $s = j - i$. Note that $1 \leq s \leq 2m-1$. Define the following walks:

$$W_1 \text{ is } w_i, w_{i+1}, \ldots, w_j, \quad \text{and} \quad W_2 \text{ is } w_0, w_1, \ldots, w_i, w_{j+1}, w_{j+2}, \ldots, w_{2m+1}.$$

(That is, $W_1$ consists of the $s+1$ vertices of $W$ from $w_i$ to $w_j$; and $W_2$ is formed from $W$ by deleting the $s-1$ vertices from $w_{i+1}$ to $w_{j-1}$.)
Observe that $W_1$ is a closed walk (since it starts at $w_i$ and ends at $w_j = w_i$. In addition, $W_2$ is also a walk, because $w_i = w_j$ is adjacent to $w_{j+1}$; and $W_2$ is also closed, since it starts and ends at $w_0 = w_{2m+1}$. In addition, $W_1$ lists $s+1$ vertices and hence has length $s$, while $W_2$ lists $(2m+1)-(s-1) = 2m+2-s$ vertices, and hence has length $2m+1-s$.
If $s$ is odd, then $W_1$ is a closed walk of odd length $s < 2m+1$, so by the inductive hypothesis, $W_1$ and hence $W$ contains an odd-length cycle.
Otherwise, $s$ is even, in which case $2m+1-s$ is odd (and strictly smaller than $2m+1$). Thus, $W_2$ is a closed walk of odd length $2m+1-s < 2m+1$, so by the inductive hypothesis, $W_2$ and hence $W$ contains an odd-length cycle. QED

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6. (7 points) Let $G$ be a graph of order $n$ and size $t$. Let $\overline{G}$ be the complement graph of $G$. (See the textbook, Section 1.1.3, item 3.) Find the order and size of $\overline{G}$.

**Solution**. Since $V(\overline{G}) = V(G)$ by definition, the order of $\overline{G}$ is also $n$.

As we saw in problem 1 of this assignment, there are $\binom{n}{2}$ possible edges between vertices of $V(G)$. Together, $E(G)$ and $E(\overline{G})$ consist of all the possible edges, with no repeats, and hence

$$|E(G)| + |E(\overline{G})| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Since $|E(G)| = t$, it follows that the size of $\overline{G}$ is

$$|E(\overline{G})| = \frac{n(n-1)}{2} - t.$$