Homework #9

Due **Wednesday, November 4** in Gradescope by **11:59 pm ET**

READ Textbook Sections 1.6.2–1.6.4, 1.7.1, and start 1.7.2.

WRITE AND SUBMIT solutions to the following problems.

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1. (12 points) Textbook Section 1.6.2, #5:
For any integer $n \geq 2$, prove that the only graph G of order n for which $\chi(G) = n$ is the complete graph K_n.
That is, prove that for any $n \geq 2$ and any graph G of order n, we have $\chi(G) = n$ if and only if G is complete.

2. (20 points) Textbook Section 1.6.2, #6:
Recall that $\alpha(G)$ and $\chi(G)$ are the independence number and chromatic number of G, respectively. Prove that for any $n \geq 1$ and any graph G of order n, we have

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G).$$

3. (14 points) Textbook Section 1.6.4, #4:
Prove that for any tree T of order n, the chromatic polynomial of T is $c_T(k) = k(k-1)^{n-1}$.
(*Suggestion*: Use Theorem 1.48 and induction on n.)

4. (16 points) Prove that for any $n \geq 3$, the chromatic polynomial of the cycle graph C_n is $c_{C_n}(k) = (k-1)\big((k-1)^{n-1} + (-1)^n\big)$.
(*Suggestion*: Use the result of Problem 3 above, Theorem 1.48, and induction on n.)

5. (14 points) Let $n \geq 2$, and let e be any edge of the complete graph K_n. Prove that K_n/e is isomorphic to K_{n-1}.

6. (12 points) For any $n \geq 2$, let $H = K_n - e$ be the graph K_n with one edge removed. Prove that the chromatic polynomial of H is is $c_H(k) = k(k-1)\cdots(k-n+3)(k-n+2)^2$.
(*Suggestion*: Don't use induction. Instead, use the result of Problem 5 above, Theorem 1.48, and the known formula for the chromatic polynomial of K_n.)

(problems continue next page)

7. (12 points) Let $n \geq 1$ be a positive integer, and let $G = K_{2,n}$. Prove that the chromatic polynomial of G is $c_G(k) = k(k-1)\big((k-2)^n + (k-1)^{n-1}\big)$.

(*Suggestion*: Don't use induction, and don't use Theorem 1.48. Instead, for the two vertices a, b on the 2-side of $G = K_{2,n}$, first count how many proper k-colorings of G assign the same color to a and b, and then count how many assign different colors to them.)

Optional Challenges (do NOT hand in): Textbook Section 1.6.2, Problem 8; Section 1.6.3, Problem 5

Questions? You can ask in:

Class: TuTh 8:35–9:50am

My office hours: Mon 1:45–3:15pm, Tue 1:45–3:15pm, and Fri, 1:00–2:00pm

Megan's Math Fellow office hours (SMUD 206):

Mondays, 7:30–9:00pm, and
Wednesdays, 7:30–9:00pm.

The QCenter also offers Graph Theory help:

Tim St. Onge (SCCE D111):

Mon–Fri, 10am–2pm
By appointment only (via Penji calendar).

Ris Paulino (SMUD 208):

Mon/Wed/Fri	10:00am–noon and 1:00pm–2:00pm
Tue/Thu	noon–3:00pm

or by appointment later in the afternoons

Also, you may email me any time at `rlbenedetto@amherst.edu`