Math 280, Fall 2026 Professor Rob Benedetto

Homework #4

Due **Wednesday, September 23** in Gradescope by **11:59 pm ET**

READ Textbook Sections 1.3.1 and 1.3.2

WATCH Video 13: Some Miscellaneous Tree Results (12:43)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (8 points) Textbook, Section 1.2.2, #3:
Let G be a graph with $V(G) = \{v_1, \dots, v_n\}$ and with adjacency matrix A. For each $j = 1, \dots, n$, prove that the (j, j) entry of A^2 is $\deg(v_j)$.

Problem 2. (14 points) Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, and let G be the graph with adjacency matrix A.

(a) Compute A^2 and A^3.

(b) How many walks are there in G from vertex 1 to vertex 2 of length exactly 3?

(c) Find the radius and the diameter of G.

(d) Draw the graph G.

Problem 3. (14 points) Textbook Section 1.3.1, #1:
Draw all unlabeled trees of order 7.
(More precisely: Find a set of trees of order 7 so that *every* tree of order 7 is isomorphic to one in your set, and so that no two in your set are isomorphic to each other.)

(*Hint*: there are 11 of them. Careful not to draw the same one twice in a different way! You don't need to give a formal proof that your set is complete; just draw 11 truly different trees of order 7. Make sure to draw **clearly**; unclear graphs will be marked wrong.)

Problem 4. (5 points) Textbook Section 1.3.1, #3:
Let T be a tree of order $n \geq 2$. Prove that T is bipartite.

(*Hint*: Do we know any theorems about when a graph is bipartite?)

Problem 5. (6 points) Textbook Section 1.3.2, #2:
Let T be a tree that has an even number of edges. Prove that at least one vertex of T has even degree.

(problems continue on next page)

Problem 6. (18 points) Textbook Section 1.3.2, (part of) #5:
Let T be a tree, and let $u, v \in V(T)$. Prove that there is *exactly one* path from u to v.

Problem 7. (14 points) Textbook Section 1.3.2, (part of) #6:
Let T be a tree, and let $u, v \in V(T)$ be two distinct vertices that are *not* adjacent. Define a new graph G with the same vertex set $V(G) = V(T)$ but with one extra edge $e = uv$. That is, $E(G) = E(T) \cup \{e\}$, where the new edge e runs between u and v.
Prove that the new graph G has exactly one cycle.

(*Suggestion*: For the uniqueness part of the proof, use the result of the previous problem.)

Problem 8. (10 points) Let T be a tree of order $n \geq 2$, and suppose that none of the vertices of T have degree 2. Prove that T has (strictly) more than $n/2$ leaves.

(*Suggestion*: Give names to the set of leaves of T and the set of vertices of T that are not leaves. What can you say about the vertices in the second of these two sets?)

Optional Challenges (do NOT hand in):
Textbook Section 1.2.2, Problem 5; Section 1.3.2, Problems 10, 12

Questions? You can ask in:

Class: TuTh 8:35–9:50am

My office hours: Mon 1:45–3:15pm, Tue 1:45–3:15pm, and Fri, 1:00–2:00pm

Megan's Math Fellow office hours (SMUD 206):

Mondays, 7:30–9:00pm, and
Wednesdays, 7:30–9:00pm.

Also, you may email me any time at `rlbenedetto@amherst.edu`