Math 280, Fall 2026

Solutions to Practice Problems 2

1. Use Prüfer's method to draw and label the trees with Prüfer sequences $1, 2, 3, 2, 1$ and $3, 2, 1, 2, 3$.

Solutions. Call the first sequence $\sigma_0 = 1, 2, 3, 2, 1$. Since it has 5 entries, the corresponding tree has $n = 7$ vertices, so let $S_0 = \{1, 2, 3, 4, 5, 6, 7\}$.

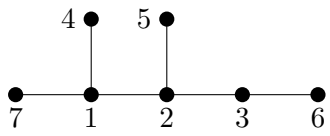
$\boxed{i = 0}$ The smallest $j \in S_0$ not in σ_0 is 4, and the first entry in σ_0 is 1. so we add the edge $\boxed{1\text{–}4}$ to make T_1. Let $\sigma_1 = 2, 3, 2, 1$ and $S_1 = \{1, 2, 3, 5, 6, 7\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in σ_1 is 5, and the first entry in σ_1 is 2. so we add the edge $\boxed{2\text{–}5}$ to make T_2. Let $\sigma_2 = 3, 2, 1$ and $S_2 = \{1, 2, 3, 6, 7\}$.

$\boxed{i = 2}$ The smallest $j \in S_2$ not in σ_2 is 6, and the first entry in σ_2 is 3. so we add the edge $\boxed{3\text{–}6}$ to make T_3. Let $\sigma_3 = 2, 1$ and $S_3 = \{1, 2, 3, 7\}$.

$\boxed{i = 3}$ The smallest $j \in S_3$ not in σ_3 is 3, and the first entry in σ_3 is 2. so we add the edge $\boxed{2\text{–}3}$ to make T_4. Let $\sigma_4 = 1$ and $S_4 = \{1, 2, 7\}$.

$\boxed{i = 4}$ The smallest $j \in S_4$ not in σ_4 is 2, and the first entry in σ_4 is 1. so we add the edge $\boxed{1\text{–}2}$ to make T_5. Let $\sigma_5 = \varnothing$ and $S_5 = \{1, 7\}$. Since σ_5 is empty, we add the one more edge from S_5, namely $\boxed{1\text{–}7}$ yielding the following final graph:



Call the second sequence $\sigma_0 = 3, 2, 1, 2, 3$. Since it has 5 entries, the corresponding tree has $n = 7$ vertices, so let $S_0 = \{1, 2, 3, 4, 5, 6, 7\}$.

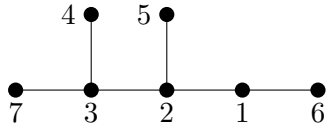
$\boxed{i = 0}$ The smallest $j \in S_0$ not in σ_0 is 4, and the first entry in σ_0 is 3. so we add the edge $\boxed{3\text{–}4}$ to make T_1. Let $\sigma_1 = 2, 1, 2, 3$ and $S_1 = \{1, 2, 3, 5, 6, 7\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in σ_1 is 5, and the first entry in σ_1 is 2. so we add the edge $\boxed{2\text{–}5}$ to make T_2. Let $\sigma_2 = 1, 2, 3$ and $S_2 = \{1, 2, 3, 6, 7\}$.

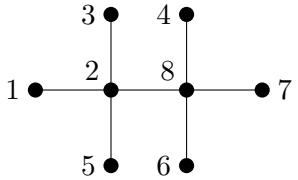
$\boxed{i = 2}$ The smallest $j \in S_2$ not in σ_2 is 6, and the first entry in σ_2 is 1. so we add the edge $\boxed{1\text{–}6}$ to make T_3. Let $\sigma_3 = 2, 3$ and $S_3 = \{1, 2, 3, 7\}$.

$\boxed{i = 3}$ The smallest $j \in S_3$ not in σ_3 is 1, and the first entry in σ_3 is 2. so we add the edge $\boxed{1\text{–}2}$ to make T_4. Let $\sigma_4 = 3$ and $S_4 = \{2, 3, 7\}$.

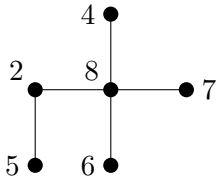
$\boxed{i = 4}$ The smallest $j \in S_4$ not in σ_4 is 2, and the first entry in σ_4 is 3. so we add the edge $\boxed{2\text{–}3}$ to make T_5. Let $\sigma_5 = \varnothing$ and $S_5 = \{3, 7\}$. Since σ_5 is empty, we add the one more edge from S_5, namely $\boxed{3\text{–}7}$ yielding the following final graph:



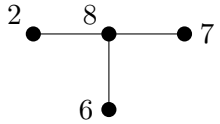
2. Use Prüfer's method to find the Prüfer sequence of this tree:



Solution.
$\boxed{i=0}$ The smallest-labelled leaf of $T = T_0$ is 1, so we record its neighbor $\boxed{2}$ and let $T_1 = T_0 - 1$.
$\boxed{i=1}$ The smallest-labelled leaf of T_1 is 3, so we record its neighbor $\boxed{2}$ and let $T_2 = T_1 - 3$, which is:



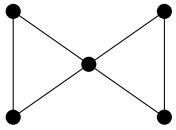
$\boxed{i=2}$ The smallest-labelled leaf of T_2 is 4, so we record its neighbor $\boxed{8}$ and let $T_3 = T_2 - 4$.
$\boxed{i=3}$ The smallest-labelled leaf of T_3 is 5, so we record its neighbor $\boxed{2}$ and let $T_4 = T_3 - 5$, which is:



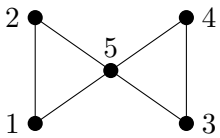
$\boxed{i=4}$ The smallest-labelled leaf of T_4 is 2, so we record its neighbor $\boxed{8}$ and let $T_5 = T_4 - 2$.
$\boxed{i=5}$ The smallest-labelled leaf of T_5 is 6, so we record its neighbor $\boxed{8}$ and let $T_6 = T_5 - 6$.
$\boxed{i=6}$ We are down to a two-vertex tree, so we stop.
Reading off what we recorded, the Prüfer sequence is $\boxed{2,2,8,2,8,8}$

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3. Use the Matrix Tree Theorem to compute the number of spanning trees of this graph:



Solution. Label the graph as follows:



So the Laplacian is $\Delta = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$ and hence, computing the (5,5)-cofactor (which I chose to preserve as many zeros as possile), the Matrix Tree Theorem says there

are

$$(-1)^{5+5}\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

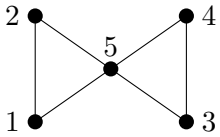
$$= 4\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4(4-1) - (4-1) = 12 - 3 = \boxed{\text{9 spanning trees}}$$

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4. Let G be the same graph as in problem 3 above.
   (a) Is G Eulerian? Prove your answer.
   (b) Is G Hamiltonian? Prove your answer.
   (c) What is the independence number of G? Prove your answer.
   (d) Is G claw-free? Prove your answer.
   (e) Is G C_3-free? Prove your answer.

Solutions. (a): Yes, $\boxed{G \text{ is Eulerian}}$ by a Theorem, since it is connected, and every vertex has even degree (the degree sequence is 4,2,2,2,2).

(b): No, $\boxed{G \text{ is } \textbf{not} \text{ Hamiltonian}}$. Label the graph as follows:

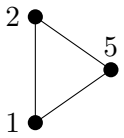


If G had a Hamiltonian cycle, we can consider it as starting anywhere, and going in either direction. Thus, we may consider it as starting at vertex 1, and then going to vertex 5. If it next goes to 2, then since it cannot repeat 5, it can never get to 3.
Therefore, the cycle must begin either $1, 5, 3$ or $1, 5, 4$. But then it can never get to 2, since it cannot repeat 5. Hence we have a contradiction, proving our claim.

(c): The independence number is $\boxed{\alpha(G) = 2}$
The set of vertices $\{1, 3\}$ is pairwise nonadjacent, verifying that $\alpha(G) \geq 2$.
To see the reverse inequality, suppose there were a set S of three or more nonadjacent vertices Then none of those vertices can be vertex 5, since 5 is adjacent to all other vertices. But since vertices $1, 2$ are adjacent, S can contain only one of $1, 2$. Similarly, S can contain only one of $3, 4$. Thus, S contains at most two elements, a contradiction. This proves $\alpha(G) \leq 2$ and hence $\alpha(G) = 2$.

(d): Yes, $\boxed{G \text{ is claw-free}}$
Given any set S of four vertices, we have two cases:
Case 1: $5 \notin S$. Then $S = \{1, 2, 3, 4\}$, and the induced graph $\langle S \rangle$ consists of two edges 12 and 34, which is not a claw.
Case 2: $5 \in S$. Then after rotating or flipping the graph over, we may assume that $1 \notin S$, i.e., $S = \{2, 3, 4, 5\}$. Then the induced graph $\langle S \rangle$ contains the cycle $3, 4, 5, 3$ and hence is not a claw.

(e): No, $\boxed{G \text{ is not } C_3\text{-free}}$
Let $S = \{1, 2, 5\}$. Then the induced graph $\langle S \rangle$ is



which is a copy of C_3.

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5. Suppose G is Hamiltonian and $S \subseteq V(G)$ be a set of $m \geq 1$ vertices. Prove that $G - S$ has at most m connected components.

Proof. Pick $y \in S$. By hypothesis, G has a Hamiltonian cycle, which may think of as starting and ending at y. Thus, we write the cycle as

$$y = x_1, x_2, \ldots, x_n, x_1 = y$$

where the n vertices $x_1, \ldots, x_n$ are all distinct, and account for all vertices of $V(G)$.
Write $S = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$ where $1 = j_1 < j_2 < \cdots < j_m \leq n$ are $m = |S|$ distinct indices between 1 and n, with $j_1 = 1$, so that $x_{j_1} = x_1 = y$.
Then for each $i = 1, \ldots, m-1$, the walk

$$x_{j_i+1}, x_{j_i+2}, \ldots, x_{j_{(i+1)}-1}$$

is a path in $G - S$, since the consecutive vertices are adjacent in G, and no vertices of S appear. So all of these vertices from $j_i + 1$ to $j_{(i+1)} - 1$ lie in the same connected component H_i of G. Similarly, the walk

$$x_{j_m+1}, x_{j_m+2}, \ldots, x_n$$

is also a path in $G - S$, so the vertices from $j_m + 1$ to n lie in a single component H_m. (Note that it is possible that $H_i = H_\ell$ for some $i \neq \ell$, if the vertices are connected by other edges; it is also possible that some H_i is empty, if $j_i + 1 = j_{i+1}$.)
These m paths together account for all the vertices of $G - S$. Thus, the (up to) m components $H_1, \ldots H_m$ account for all of $G - S$. QED

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6. Suppose G is Hamiltonian. Prove that $\kappa(G) \geq 2$.

Proof. G has a Hamiltonian cycle containing all the vertices of $V(G)$. Thus, G is connected, since for any two vertices $u, v \in V(G)$, both u and v lie along the cycle, so we can walk along the cycle to get from u to v. That is, $\kappa(G) \geq 1$.
To prove that $\kappa(G) \geq 2$, given any $y \in V(G)$, we must show that $G - y$ is connected. We may view the Hamiltonian cycle as starting and ending at y, so the cycle is:

$$y = x_1, x_2, \ldots, x_n, x_1 = y.$$

Thus, $x_2, \ldots, x_n$ is a path in $G - y$ that uses every vertex. In particular, for any $u, v \in V(G - y)$, we can walk along this path from u to v.
Thus, $G - y$ is connected for any $y \in V(G)$ and hence $\kappa(G) \geq 2$ QED

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7. Give an example of a graph G of order at least 4 that is traceable (i.e., that has a Hamiltonian path) and has $\kappa(G) < 2$.

Proof. Let $G = P_4$ (or P_n for any $n \geq 4$), with the vertices numbered $1, 2, 3, 4$ from one end to the other. Then $1, 2, 3, 4$ is a Hamitonian path.
However, $G-3$ consists of two separate components, namely $\{1, 2\}$ and $\{4\}$, so $\kappa(G) = 1 < 2$.

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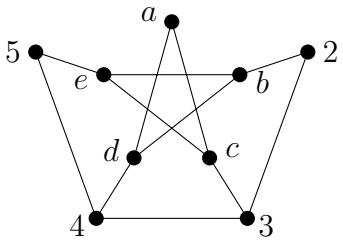
8. Prove that the graph $K_{4,4}$ is Eulerian but not planar.

Proof. The graph is connected, and every vertex of $K_{4,4}$ has degree 4, which is even. So by a Theorem, $K_{4,4}$ is Eulerian.
However, selecting 3 vertices from each side of this bipartite graph, $K_{4,4}$ contains a subgraph isomorphic to $K_{3,3}$. If $K_{4,4}$ were planar, then every subgraph would be planar, and hence $K_{3,3}$ would be planar, a contradiction. Thus, $K_{4,4}$ is not planar. QED

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9. Let G be the Petersen graph, and let $v \in V(G)$. (You may assume that v is the vertex at the top of the outer pentagon.) Prove that $G - v$ is Hamiltonian.

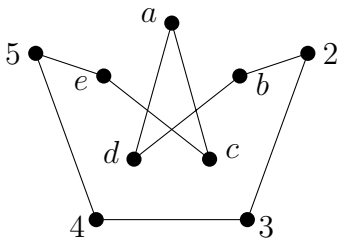
Proof. Here is $G - v$:



Then we can see that

$$2, 3, 4, 5, e, c, a, d, b, 2$$

is a Hamiltonian cycle, which looks like this:



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10. Let G be an Eulerian graph. Prove that the line graph $L(G)$ is Hamiltonian.

Proof. By hypothesis, G has an Eulerian circuit

$$x_1, x_2, \ldots, x_m, x_{m+1} = x_1,$$

which means that the edges are

$$e_1 = x_1x_2,\ e_2 = x_2x_3,\ \ldots,\ e_{m-1} = x_{m-1}x_m,\ e_m = x_mx_1.$$

That is, $e_1, \ldots, e_m$ are all distinct, and $E(G) = \{e_1, \ldots, e_m\}$.
Note that $m \geq 3$. Indeed, if $m = 0$, then since G has order at least 2 it would be disconnected and hence have no circuits. If $m = 1$, then after using the one edge, one could not return to start to complete the circuit. And if $m = 2$, then since the two edges are distinct, then

following one and then the other cannot return one to the starting vertex. This verifies our claim that $m \geq 3$.
We have $V(L(G)) = E(G) = \{e_1, \ldots, e_m\}$ by definition. Moreover, in $L(G)$, for each $i = 1, \ldots, m-1$, the vertices e_i and e_{i+1} are adjacent, because in G they are edges that share the vertex v_{i+1}. Similarly, the vertices e_m and e_1 of $L(G)$ are adjacent, because in G they are edges that share the vertex v_1. Thus,

$$e_1, e_2, \ldots, e_m, e_1$$

is a cycle in $L(G)$, since it a closed walk (each vertex is adjacent to the next), it has no repeated vertices, and $m \geq 3$ as noted above. In addition, it uses all the vertices of $L(G)$. Thus, $L(G)$ has a Hamiltonian cycle. QED

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11. Let G be a 4-regular graph (i.e., each vertex has degree 4) that is connected and has order n. If G is planar, prove that the number of regions in a planar representation of G is $r = n + 2$.

Proof. Let q be the number of edges of G. By an old theorem,

$$2q = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} 4 = 4n,$$

and hence $q = 2n$.
On the other hand, since G is connected and planar, Euler's formula gives $n - q + r = 2$. Thus,

$$r = 2 - n + q = 2 - n + 2n = n + 2,$$

as desired. QED

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12. Let e_1 and e_2 be any two edges of K_6, and let $G = K_6 - e_1 - e_2$. Prove that G is not planar.

Proof. (**Method 1**): We consider two cases.
Case 1. e_1 and e_2 share a vertex v. Then $G' = K_6 - v$ is a subgraph of G, since neither edge e_1 nor e_2 belongs to $E(G')$. However, $G' = K_6 - v$ is isomorphic to K_5, since it has 5 vertices, any two of which are connected by an edge. If G were planar, then its subgraph G' would be planar, too, and hence K_5 would be planar, a contradiction.
Case 2. e_1 and e_2 do not share any vertices. We may label the vertices of e_1 as $1, 2$ and the vertices of e_2 as $5, 6$. Then for each $i = 1, 2, 3$ and each $j = 4, 5, 6$, the edge ij belongs to $E(G)$. Thus, G contains a subgraph G'' isomorphic to $K_{3,3}$, with one set of 3 vertices being $\{1, 2, 3\}$, and the other set being $\{4, 5, 6\}$. (Just delete the remaining edges 23 and 13 among $\{1, 2, 3\}$, and the remaining edges 45 and 46 among $\{4, 5, 6\}$.) If G were planar, then its subgraph G'' would be planar, too, and hence $K_{3,3}$ would be planar, a contradiction. QED

(**Method 2**): The number of vertices of K_6 is 6, and the number of edges is $\binom{6}{2} = 15$. Thus, setting $n = |V(G)|$ and $q = |E(G)|$, we have Thus, $n = |V(K_6)| = 6$, and $q = |E(K_6)| - 2 = 13$.

If G were planar, then by a Theorem [namely Theorem 1.33], we would have $q \leq 3n - 6$. But $q = 13 \not\leq 12 = 3n - 6$. So G is not planar. QED

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13. Let G be a Hamiltonian graph, let $m \geq 1$, and let $a, b, c, x_1, \ldots, x_m \in V(G)$ be $m + 3$ distinct vertices. Suppose that the neighborhood of a is

$$N(a) = \{b, c, x_1, \ldots, x_m\}$$

and that $\deg(b) = \deg(c) = 2$. Let $G' = G - e_1 - e_2 - \cdots - e_m$, where each e_i is the edge from a to x_i. Prove that G' is Hamiltonian.

Proof. By hypothesis, G has a Hamiltonian cycle, which we can consider as starting anywhere, and going in either direction. In particular, we may assume it starts at b, and since b has only two neighbors, we may assume the second vertex in the path is a.
Suppose (towards a contradiction) that the third vertex in the path is **not** c. Let y be the unique other neighbor of c. Then when the cycle eventually arrives at c, the preceding vertex must be y, since we cannot reuse a. But then upon arriving at c, we cannot continue, as both of its neighbors, a and y, have already been used. That is, the cycle looks like:

$$b, a, \ldots, y, c, ?$$

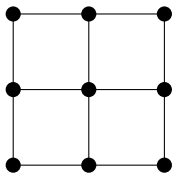
but there is nothing that can go in the position marked by the ?. Thus, we have our desired contradiction, and the third vertex in the path **must** be c.
That is, the Hamiltonian cycle must be of the form

$$b, a, c, \ldots, b,$$

with the vertices $x_1, \ldots, x_m$ all appearing in the middle of the cycle somewhere. Since vertex a already appeared near the start, it cannot appear again in the "..." portion of the above cycle, so none of the edges $e_1, \ldots, e_m$ is used by the above cycle. Thus, the above cycle is also a Hamiltonian cycle for the graph $G' = G - e_1 - e_2 - \cdots - e_m$. QED

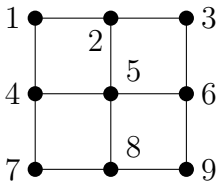
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14. Let G be the following graph:

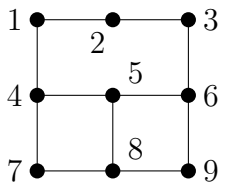


Use the previous problem to prove that G is not Hamiltonian.

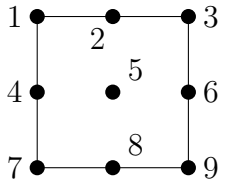
Proof. Suppose G were Hamiltonian. Label the graph as follows:



Note that vertex 2 has neighbors $1, 3$, both of which have degree 2. By the previous problem, since G is Hamiltonian, so is the subgraph obtained by removing the edge 25, i.e.:



Similarly, vertex 6 has neighbors $3, 9$, both of which have degree 2, so we may remove edge 56 and still be Hamiltonian. By similar logic, we may then remove edge 58 (since 8 has degree-2 neighbors 7 and 9), and then the edge 45 (since 4 has degree-2 neighbors 1 and 7). Thus, the original assumption that G is Hamiltonian implies that the following graph G' is Hamiltonian:



However, this graph is disconnected, and in particular no cycle can reach the vertex 5. That is, G' cannot be Hamiltonian, and hence the original graph G is not Hamiltonian. QED

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15. Let $r \geq 1$ be a positive integer, and G be a connected graph that is regular of degree r. [That is, all vertices of G have degree r.] Prove that the line graph $L(G)$ is Eulerian.

Proof. We first claim that $L(G)$ is connected. Given two vertices $e_1, e_2 \in V(L(G)) = E(G)$, write $e_i = v_i w_i$ for $i = 1, 2$. Since G is connected, there is a $v_1 - v_2$ path in G, given by $v_1 = x_0, x_1, \ldots, x_k = v_2$. For each $j = 1, \ldots, k$, let $f_j \in E(G)$ be the edge from x_{j-1} to x_j. As vertices in $L(G)$, note that each f_j is adjacent to f_{j+1}, since in G they share the vertex x_j. Similarly, f_1 either coincides with or is adjacent to e_1 (since they both share v_1), and f_k either coincides with or is adjacent to e_2 (since they both share v_2). Thus, $e_1, f_1, f_2, \ldots, f_k, e_2$ — where we omit f_1 in this list if $f_1 = e_1$, and we omit f_k if $f_k = e_2$ — is a walk in $L(G)$ from e_1 to e_2, proving our claim.

Given any vertex $e \in V(L(G)) = E(G)$, write $e = uv$ with $u, v \in V(G)$. Since $\deg_G(u) = r$, there are exactly r distinct vertices $x_1, \ldots, x_r \in V(G)$ adjacent to u in G; without loss, $v = x_r$. Similarly, since $\deg_G(v) = r$, there are exactly r distinct vertices $y_1, \ldots, y_r \in V(G)$ adjacent to u in G; without loss, $v = y_r$. [Note that while the x_i's are all distinct, and the y_i's are all distinct, it is possible that some x_i coincides with some y_j.]

For each $i = 1, \ldots, r$, define $e_i = ux_i \in E(G)$ and $e_i' = vy_i \in E(G)$. Then $e_r = e_r' = uv = e$, but otherwise the edges $e_1, \ldots, e_{r-1}$ and $e_1', \ldots, e_{r-1}'$ are all distinct, since for $i < r$, the e_i's are incident with u but not v, and the e_i''s are incident with v but not u. In addition, viewed as vertices in $L(G)$, each e_i is adjacent to $e = uv$ for $i < r$, since they both share vertex u in G; and similarly each e_i' is adjacent to $e = uv$ for $i < r$. Moreover, no other edge of $E(G)$ is incident with u or v, so no other vertices of $L(G)$ are adjacent to $e \in V(L(G))$.
Thus, there are exactly $2(r-1)$ vertices of $L(G)$ adjacent to e, so $\deg(e) = 2(r-1)$ is even.

We have shown that $L(G)$ is connected, and every vertex has even degree. Therefore, by a theorem, $L(G)$ is Eulerian. QED

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16. For any positive integers $a, b, c \geq 1$, the *complete tripartite graph* $K_{a,b,c}$ is the following graph. The vertex set is $A \cup B \cup C$, where A, B, C are pairwise disjoint sets of orders a, b, c, respectively. For any $a \in A$, $b \in B$, and $c \in C$, there are edges from a to b, from b to c, and from a to c; but there are no edges between any two vertices of A, any two vertices of B, or any two vertices of C.
Prove that $K_{a,b,c}$ is Eulerian if and only if a, b, c are either all even or all odd.

Proof. We claim that $G = K_{a,b,c}$ is connected. To see this, given vertices x, y, first suppose that x and y lie in different sets among A, B, C; without loss, we have $x \in A$ and $y \in B$. Then $xy \in E(G)$, so x, y is a path from x to y. Otherwise, x, y live in the same set; without loss, we have $x, y \in A$. Pick $v \in B$, which exists since $|B| = b \geq 1$. Then $xv, yv \in E(G)$, so x, v, y is a path from x to y, proving our claim.

Each $v \in A$ has edges to each of the vertices in $B \cup C$, and no other incident edges, so $\deg(v) = b + c$. Similarly, each $v \in B$ has $\deg(v) = a + c$, and each $v \in C$ has $\deg(v) = a + b$.

Since G is a connected, it follows by a Theorem that G is Eulerian if and only if all three of the numbers $a + b$, $b + c$, and $a + c$ are even. But $a + b$ is even if and only if a and b have the same parity, and similarly for the other pairs. Thus, G is Eulerian if and only if a, b, c are all of the same parity. QED

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17. (Section 1.4.3, #6): Let G be a graph of order n that is r-regular for some $r \geq 1$. Prove that $\alpha(G) \leq n/2$.

Proof. Let $m = \alpha(G)$, and let $S \subseteq V(G)$ be an independent set of m vertices.
For any vertex $v \in S$, let E_v be the set of all r edges of G incident with v.
For any two distinct vertices $v, w \in S$, there is no edge vw in G (by definition of independent set), and hence $E_v \cap E_w = \varnothing$. Thus,

$$|E(G)| \geq \left| \bigcup_{v \in S} E_v \right| = \sum_{v \in S} |E_v| = \sum_{v \in S} r = mr.$$

On the other hand, since G is r-regular, the first theorem of graph theory tells us that

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} r = nr.$$

Combining these two facts, we have

$$2mr \leq 2|E(G)| = nr.$$

Dividing by $2r > 0$, we obtain $m \leq n/2$. QED

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18. (Section 1.5.2, #2): Let G be a connected planar graph of order at most 11. Prove that $\delta(G) \leq 4$.

Proof. Suppose, towards contradiction, that $\delta(G) > 4$. Thus, we have $\deg(v) \geq 5$ for every vertex $v \in V(G)$. Hence, writing n for the order and q for the size of G, we have

$$5n \leq \sum_{v \in V(G)} \deg(v) = 2q \leq 2(3n - 6) = 6n - 12,$$

where the first equality is by the first theorem of the course, and the second inequality is by a theorem about planar graphs.
Thus, $n \geq 12$, a contradiction. So $\delta(G) \leq 4$. QED

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19. Give an example of a graph G with an edge e such that G is claw-free but $G - e$ is *not* claw-free.

Solution. Consider the following graph G, with edge e, and the resulting graph $G - e$



G $\quad$ $G - e$

Then G is claw-free, since the only way to choose four vertices of G is to choose *all* vertices of G, so that the resulting induced subgraph is G itself, which is not a claw.
However, $G - e$ is itself a claw, so $G - e$ is *not* claw-free. QED

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20. Give an example of a graph G with an edge e such that $G - e$ is claw-free but G is *not* claw-free.

Solution. Consider the following graph G, with edge e, and the resulting graph $G - e$



G $\quad$ $G - e$

Then G is *not* claw-free, since G itself is a claw.
However, the only way to choose four vertices of $G - e$ is to choose *all* vertices of $G - e$, so that the resulting induced subgraph is $G - e$ itself, which is not a claw. Hence, $G - e$ *is* claw-free. QED

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21. Give an example of a graph G with a vertex v such that $G - v$ is claw-free but G is *not* claw-free.

Solution. Consider the following graph G, with vertex v, and the resulting graph $G - v$



G $\quad$ $G - v$

Then G is *not* claw-free, since G itself is a claw.
However, $G - v$ has only three vertices, and hence no subgraphs isomorphic to the claw, which has four vertices. Therefore, $G - v$ *is* claw-free. QED

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22. Let G be a claw-free graph, and let $v \in V(G)$. Prove that $G - v$ is also claw-free.

Proof. Given a set $S = \{w_1, w_2, w_3, w_4\} \in V(G - v)$ of four distinct vertices in $G - v$, then we have $w_i \in V(G)$ with v, w_1, w_2, w_3, w_4 all distinct.
For any $i, j \in \{1, 2, 3, 4\}$, then, we have w_i adjacent to w_j in G if and only if w_i is adjacent to w_j in $G - v$, since the only edges in $E(G)$ removed to make $E(G - v)$ are those incident

with v, which is distinct from both w_i and w_j.
Thus, the subgraph H induced by S in G is the same as the subgraph H' induced by S in $G - v$.
By hypothesis, H is not a claw, since G is claw-free. Thus, $H' = H$ is not a claw.
Since S was arbitrary, we have shown that $G - v$ is claw-free. QED