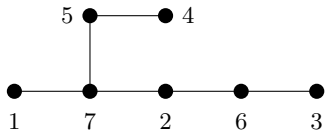
Math 280, Fall 2026

Solutions to Midterm Exam 2

1. **(15 points, 2 parts)** Don't forget to **show your steps** as you use Prüfer's method to:

1a. Draw and label the tree with Prüfer sequence 2,5,5,1,2

1b. Find the Prüfer sequence of this labelled tree:



Solutions. (a): Call the sequence $\sigma_0 = 2, 5, 5, 1, 2$. Since it has 5 entries, the corresponding tree has $n = 7$ vertices, so let $S_0 = \{1, 2, 3, 4, 5, 6, 7\}$.

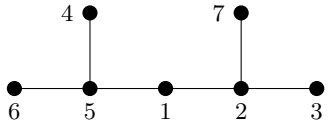
$\boxed{i = 0}$ The smallest $j \in S_0$ not in σ_0 is 3, and the first entry in σ_0 is 2. so we add the edge $\boxed{2\text{–}3}$ to make T_1. Let $\sigma_1 = 5, 5, 1, 2$ and $S_1 = \{1, 2, 4, 5, 6, 7\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in σ_1 is 4, and the first entry in σ_1 is 5. so we add the edge $\boxed{4\text{–}5}$ to make T_2. Let $\sigma_2 = 5, 1, 2$ and $S_2 = \{1, 2, 5, 6, 7\}$.

$\boxed{i = 2}$ The smallest $j \in S_2$ not in σ_2 is 6, and the first entry in σ_2 is 5. so we add the edge $\boxed{5\text{–}6}$ to make T_3. Let $\sigma_3 = 1, 2$ and $S_3 = \{1, 2, 5, 7\}$.

$\boxed{i = 3}$ The smallest $j \in S_3$ not in σ_3 is 5, and the first entry in σ_3 is 1. so we add the edge $\boxed{1\text{–}5}$ to make T_4. Let $\sigma_4 = 2$ and $S_4 = \{1, 2, 7\}$.

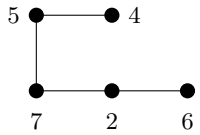
$\boxed{i = 4}$ The smallest $j \in S_4$ not in σ_4 is 1, and the first entry in σ_4 is 2. so we add the edge $\boxed{1\text{–}2}$ to make T_5. Let $\sigma_5 = \varnothing$ and $S_5 = \{2, 7\}$. Since σ_5 is empty, we add the one more edge from S_5, namely $\boxed{2\text{–}7}$ yielding the following final graph:



(b):

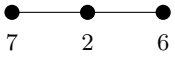
$\boxed{i = 0}$ The smallest-labelled leaf of $T = T_0$ is 1, so we record its neighbor $\boxed{7}$ and let $T_1 = T_0 - 1$.

$\boxed{i = 1}$ The smallest-labelled leaf of T_1 is 3, so we record its neighbor $\boxed{6}$ and let $T_2 = T_1 - 3$, which is:



$\boxed{i = 2}$ The smallest-labelled leaf of T_2 is 4, so we record its neighbor $\boxed{5}$ and let $T_3 = T_2 - 4$.

$\boxed{i = 3}$ The smallest-labelled leaf of T_3 is 5, so we record its neighbor $\boxed{7}$ and let $T_4 = T_3 - 5$, which is:



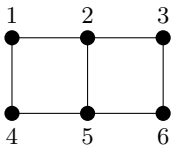
$\boxed{i = 4}$ The smallest-labelled leaf of T_4 is 6, so we record its neighbor $\boxed{2}$ and let $T_5 = T_4 - 6$.

$\boxed{i = 5}$ We are down to a two-vertex tree, so we stop.

Reading off what we recorded, the Prüfer sequence is $\boxed{7,6,5,7,2}$

---

2. **(20 points, 4 parts)** Let G be the following graph:



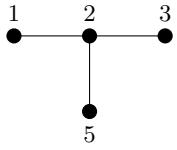
(a) Show that G is **not** claw-free.

(b) Recall that $\alpha(G)$ denotes the independence number of G. Prove that $\alpha(G) \geq 3$.

(c) Recall that $\kappa(G)$ denotes the connectivity of G. Prove that $\kappa(G) \leq 2$.

(d) Show that G **is** Hamiltonian, by producing a Hamiltonian cycle in G.

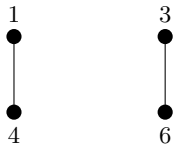
Solutions/Proofs. (a): Let $S = \{1, 2, 3, 5\}$. The induced subgraph $\langle S \rangle$ is:



which is a claw. So G is not claw-free. QED (a)

(b): Let $T = \{1, 5, 3\}$, which is an independent set of vertices of G, since no two of these three vertices are adjacent. Thus, $\alpha(G) \geq |T| = 3$. QED (b)

(c): Deleting the two vertices 2 and 5 yields the graph $G - 2 - 5$, which is:

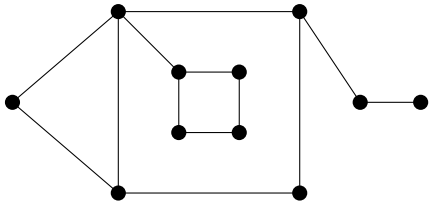


and which is disconnected. Thus, $\kappa(G) \leq 2$. QED (c)

(d): The walk 1,2,3,6,5,4,1 is a Hamiltonian cycle on G. QED (d)

---

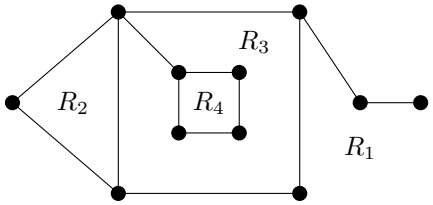
3. **(15 points, 2 parts.)** Let G be the following planar graph:



(a) Find the number of regions of G, and verify Euler's formula for G.

(b) Find the bound number $b(R)$ of each of the regions of G.

Solutions. (a): There are four regions, as follows:



Thus, in Euler's formula, we have $r = 4$.
Carefully counting vertices shows $n = 4 + 4 + 1 + 2 = 11$ (the four in the inner square, the four in the outer square, the one on the left, and the two on the right).
Carefully counting edges shows $q = 4 + 4 + 1 + 2 + 2 = 13$ (the four in the inner square, the four in the outer square, the one inner diagonal, the two on the left, and the two on the right).
Thus, we do indeed have $n - q + r = 11 - 13 + 4 = -2 + 4 = 2$.

(b): With labeling as above, we have:

$\boxed{b(R_1) = 5}$ (2 edges shared with R_2, and 3 with R_3)

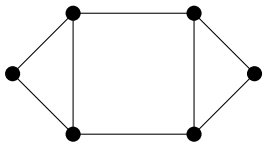
$\boxed{b(R_2) = 3}$ (2 edges shared with R_1, and 1 with R_3)

$\boxed{b(R_3) = 8}$ (3 edges shared with R_1, 2 with R_2, and 4 with R_4)

$\boxed{b(R_4) = 4}$ (all 4 edges shared with R_3)

---

4. **(16 points, 3 parts.)** Let G be the following graph:

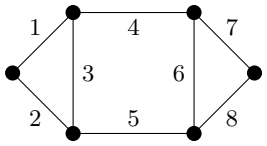


(a) Draw the line graph $L(G)$.

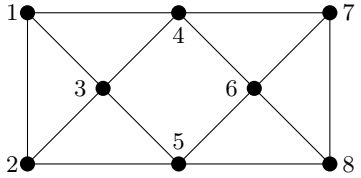
(b) Show that $L(G)$ is Hamiltonian by presenting a Hamiltonian cycle on $L(G)$.

(c) Show that the original graph G is **not** Eulerian (even though $L(G)$ is Hamiltonian).

Solutions. Label the edges of G as follows:



With that labeling, here is $L(G)$:



(b): Observe that $\boxed{1, 2, 3, 5, 6, 8, 7, 4, 1}$ is a walk in $L(G)$, since any two consecutive entries in this list are edges of G that share a vertex in G. [Equivalently, they are vertices of $L(G)$ that are adjacent in $L(G)$.]
With no repeats (except first and last), it is a cycle in $L(G)$, and it is a Hamiltonian cycle since it uses every edge of G, i.e., every vertex of $L(G)$.

(c): There are vertices of G of odd degree (in fact, four of them, the ones on the central square of G), so by Euler's Theorem, G is not Eulerian.

---

5. **(16 points)** Let $n > m \geq 1$ be (different) positive integers, and let G be the complete bipartite graph $G = K_{m,n}$. Prove that the independence number of G is $\alpha(G) = n$.

Proof. Let X and Y be the two partite sets of G, so that $|X| = m$ and $|Y| = n$, with $V(G) = X \cup Y$, with $X \cap Y = \varnothing$, and with edges from every $x \in X$ to every $y \in Y$ (and no other edges).

Then the set $Y \subseteq V(G)$ is an independent set, since there are no edges between any $y_1, y_2 \in Y$. Because $|Y| = n$, it follows that $\alpha(G) \geq n$.

Given a set $S \subseteq V(G)$ with $|S| > n$, observe that $S \not\subseteq X$ (since $|S| > n > m = |X|$), and also $S \not\subseteq Y$ (since $|S| > n = |Y|$). Therefore, there exist $u, v \in S$ with $u \notin X$ and with $v \notin Y$. That is, $u \in Y$ and $v \in X$, and therefore the edge uv **is** an edge of G, so that u and v are adjacent. Thus, S is **not** an independent set. Since this is true for all such S, we have $\alpha(G) \leq n$.

Since $\alpha(G) \geq n$ and $\alpha(G) \leq n$, we have $\alpha(G) = n$. QED

---

6. **(18 points)** Let G be a Hamiltonian graph. Prove that $\kappa(G) \geq 2$.

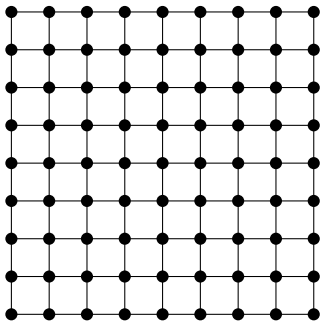
Proof.G has a Hamiltonian cycle containing all the vertices of $V(G)$. Thus, G is connected, since for any two vertices $u, v \in V(G)$, both u and v lie along the cycle, so we can walk along the cycle to get from u to v. That is, $\kappa(G) \geq 1$.

To prove that $\kappa(G) \geq 2$, given any $y \in V(G)$, we must show that $G - y$ is connected. We may view the Hamiltonian cycle as starting and ending at y, so the cycle is:

$$y = x_1, x_2, \ldots, x_n, x_1 = y.$$

Thus, $x_2, \ldots, x_n$ is a path in $G - y$ that uses every vertex. In particular, for any $u, v \in V(G - y)$, we can walk along this path from u to v.
Thus, $G - y$ is connected for any $y \in V(G)$ and hence $\kappa(G) \geq 2$ QED

OPTIONAL BONUS. (2 points.) Let G be the following graph, consisting of a 9×9 grid of vertices connected by horizontal and vertical edges:



Prove that G is *not* Hamiltonian.

Proof. Label the vertices as (i, j) with $1 \leq i, j \leq 9$. Thus, (i, j) is adjacent to $(i \pm 1, j)$ and $(i, j \pm 1)$ (omitting $i + 1$ if $i = 9$, and $i - 1$ if $i = 0$, and similarly for j).

Define $X = \{(i, j) \in V(G) \,|\, i + j \text{ is even } \}$ and $Y = \{(i, j) \in V(G) \,|\, i + j \text{ is odd } \}$.
[That is, think of G as a 9x9 checkerboard, with X squares and Y squares.]

Then clearly $V(G) = X \cup Y$ with $X \cap Y = \varnothing$, since every integer $i + j$ is either even or odd but not both.

Note that no two vertices of X are adjacent, since $(i \pm 1) + j$ and $i + (j \pm 1)$ both have opposite parity from $i + j$. Similarly, no two vertices of Y are adjacent. Thus, G is a bipartite graph.

By the Bipartite Graph Theorem, then, G contains no cycle of odd length, and hence no cycle of length 81. Since $|V(G)| = 9 \cdot 9 = 81$, it follows that there is no Hamiltonian cycle. QED