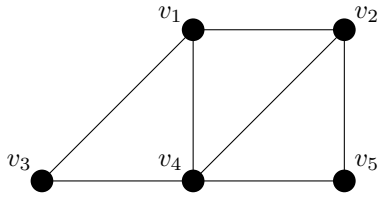
Math 280, Fall 2026 Professor Rob Benedetto

Solutions to Midterm Exam 1

1. **(18 points, 3 parts)** Let G be the following graph.



(a) Find the adjacency matrix A of the graph G.

(b) Determine the eccentricity $\text{ecc}(v_i)$ of each of the vertices of G.

(c) Find the radius, diameter, and center of G.

Solutions. (a): By inspection, we have $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

(b): v_1 is adjacent to v_2, v_3, v_4, and distance 2 from v_5 (e.g. via v_1, v_2, v_5), so $\boxed{\text{ecc}(v_1) = 2}$

v_2 is adjacent to v_1, v_4, v_5, and distance 2 from v_3 (e.g. via v_2, v_1, v_3), so $\boxed{\text{ecc}(v_2) = 2}$

v_3 is adjacent to v_1, v_4, and distance 2 from v_2 and v_5 (e.g. via v_3, v_4, v_2 and v_3, v_4, v_5), so $\boxed{\text{ecc}(v_3) = 2}$

v_4 is adjacent to all four other vertices, so $\boxed{\text{ecc}(v_4) = 1}$

v_5 is adjacent to v_2, v_4, and distance 2 from v_1 and v_3 (e.g. via v_5, v_4, v_1 and v_5, v_4, v_3), so $\boxed{\text{ecc}(v_5) = 2}$

(c): $\boxed{\text{diam}(G) = 2}$ since that is the maximum eccentricity.

$\boxed{\text{rad}(G) = 1}$ since that is the minimum eccentricity.

The $\boxed{\text{center of } G \text{ is the one vertex } v_4}$, since that is the only point attaining the minimum eccentricity.

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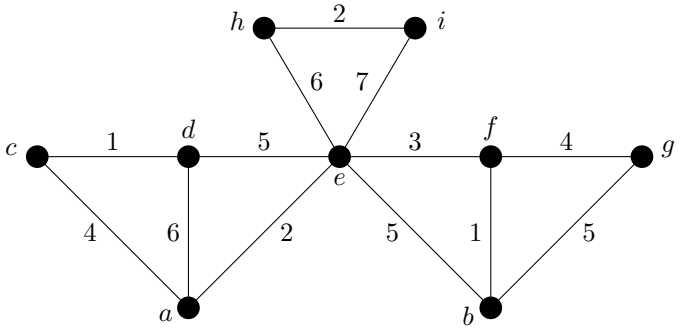
2. **(18 points)** Use Kruskal's algorithm to find a minimum spanning tree on the following weighted graph. (The letters are vertex names; the numbers are weights of edges.) As always, show your work.



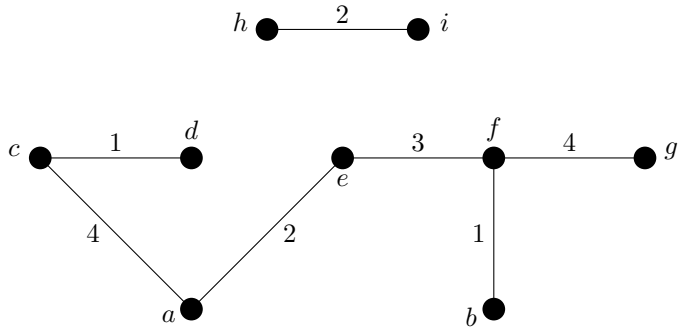
Solution.

Steps 1–2: Add the two shortest edges, cd and bf (both length 1).

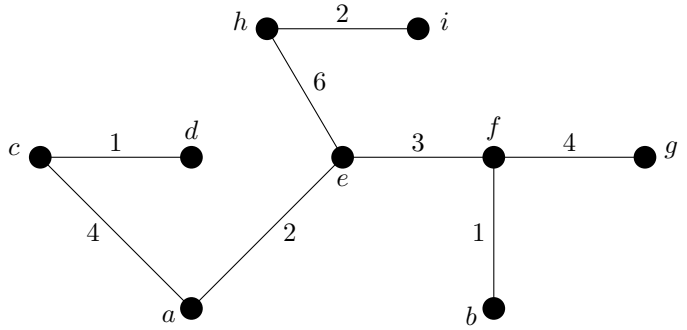
Steps 3–4: Add the next two shortest edges, ae and hi (both length 2).

Step 5: Add the next shortest edge, ef (length 3).

Steps 6–7: Add the next two shortest edges, ac and fg (both length 4). The subgraph so far now looks like this:



Step 8: The next shortest edges are de, be, and bg (all length 5), but any one of them would create a cycle, so we omit them.
Step 9: The next shortest edges are ad and eh (both length 6), but ad would create a cycle. So we add only eh, and we now have the following spanning tree, completing the algorithm:



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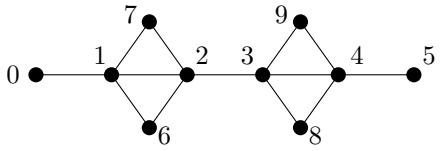
3. **(18 points, 3 parts)** Consider the following graph G, which has 10 vertices and 13 edges:



(a) Determine the degree sequence of G.

(b) Find a trail in G of maximum length, and briefly explain why no longer trail is possible.

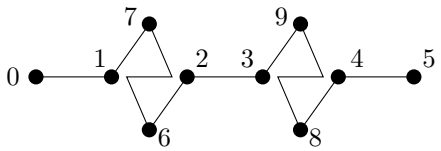
(c) What is the connectivity $\kappa(G)$? Why?

Solution. (a): By inspection, we have the following degrees:

v	0	1	2	3	4	5	6	7	8	9
deg	1	4	4	4	4	1	2	2	2	2

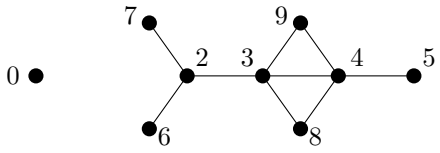
Putting them in decreasing order, the degree sequence is $\boxed{4,4,4,4,2,2,2,2,1,1}$

(b): Here is a trail of length 13: $\boxed{0,1,7,2,1,6,2,3,9,4,3,8,4,5}$
It is indeed a trail, since no edges are repeated, and it uses every single edge, as this picture shows:



And no longer trail is possible, since any longer *walk* would have to have at least 14 edges and hence repeat at least one, making it not a trail.

(c): We claim that $\boxed{\kappa(G) = 1}$
We have $\kappa(G) \geq 1$ since G is connected [e.g. the trail in (b) connects any two vertices].
And we have $\kappa(G) \leq 1$, because if we delete vertex 1, we get the following graph $G - 1$:



which is disconnected, since there is no path from vertex 0 to vertex 2 in $G - 1$, since there are no edges incident with 0.

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4. **(14 points.)** Let T be a tree of order n, and suppose that every vertex of T has degree either 1 or 3. Prove that T has exactly $1 + (n/2)$ leaves.

Proof. Let ℓ be the number of leaves of T, so that T has exactly ℓ vertices of degree 1.
By hypothesis, then, T has exactly $n - \ell$ vertices of degree 3, and no other vertices. Thus,

$$2n - 2 = 2(n-1) = 2|E(T)| = \sum_{v \in V(T)} \deg(v) = 3(n-\ell) + 1(\ell) = 3n - 2\ell,$$

where the second equality is by a theorem because T is a tree, and the third is the formula from another theorem for all graphs. Thus, $2\ell = 2 + n$, and hence $\ell = 1 + (n/2)$. QED

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5. **(22 points.)** Let T be a tree, and let $v \in V(T)$ be a vertex. Suppose that $\deg(v) = 3$. Prove that $T - v$ has a least three connected components.

Proof. Let w_1, w_2, w_3 be the distinct neighbors of v in T. It suffices to show that the vertices w_1, w_2, w_3 each lie in separate components of $T - v$. Given two such vertices, we may assume WLOG that they are w_1 and w_2.
Suppose, towards a contradiction, that w_1 and w_2 are in the same connected component of $T - v$. Then there is a path W in $T - v$ of the form

$$w_1 = x_1, x_2, \ldots, x_k = w_2.$$

Since $w_1 \neq w_2$, we have $k \geq 2$. Define a new walk W' by appending the edges $w_2 v$ and then $v w_1$ of T, i.e., given by

$$w_1 = x_1, x_2, \ldots, x_k = w_2, v, w_1.$$

Then W' is a closed walk of length $k + 1 \geq 3$. In addition, since W was a path, the vertices $x_1, \ldots, x_k$ are all distinct; and since W was a path in $T - v$, none of $x_1, \ldots, x_k$ is v. Thus, the only repeated vertex in W' is $w_1 = x_1$ itself, so that W' is a cycle. That is, T has a cycle, which is a contradiction. Thus, for any $i \neq j$, we have that w_i and w_j lie in separate components of $T - v$, and hence $T - v$ has (at least) m components. QED

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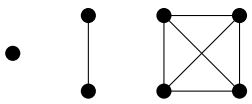
6. **(10 points.)** Give an example of a graph with exactly 3 connected components that has exactly 7 vertices and 7 edges.

Solution. Here is such a graph:



OPTIONAL BONUS. (2 points.) Let G be a connected graph that has exactly two cycles. Prove that $|E(G)| = |V(G)| + 1$, i.e., that G has exactly one more edge than it has vertices.

Proof. Let C and D be the two cycles of G. Write C as $x_1, \ldots, x_k = x_1$ and $D = y_1, \ldots, y_\ell = y_1$, with $x_1, \ldots, x_{k-1}$ all distinct, with $y_1, \ldots, y_{\ell-1}$ all distinct, and with $k, \ell \geq 4$.
[Note: the lengths of the cycles are $k - 1 \geq 3$ and $\ell - 1 \geq 3$.]

[Warning: It is possible that some x_i's might be the same as some y_j's, although that observation turns out to be irrelevant here. I just mention it as a side comment, to note that it would be incorrect to assume that the x_i's and y_j's are *all* disinct.]

Since the two cycles are different, somewhere there is an edge of C or D that is *not* an edge of the other cycle. Without loss, this is the edge $e = y_1y_2$ in cycle D.
Let $G_1 = G - e$. Then G_1 has exactly one cycle, namely C, since e was not an edge of C, and since G_1 no longer has cycle D.
In addition, we claim G_1 is connected, as we now prove. Given $v, w \in V(G_1) = V(G)$, then there is a path P from v to w in G, since G is connected. If P does not use edge e, then P is also a path from v to w in G_1, as desired. Otherwise, writing P as $v = z_1, \ldots, z_m = w$, there is some $i \in \{1, \ldots, m\}$ such that either $z_i = y_1$ and $z_{i+1} = y_2$ or else $z_i = x_2$ and $z_{i+1} = y_1$. Without loss it is the latter, i.e., $z_i = y_2$ and $z_{i+1} = y_1$. We may then define a walk W from v to w in G_1 by

$$v = z_1, z_2, \ldots, z_i = y_2, y_3, \ldots, x_k = y_1 = z_{i+1}, z_{i+2}, \ldots, z_m = w.$$

[That is, follow P from $v = z_1$ to $z_i = y_2$, then follow $D - e$ from y_2 to $y_k = y_1$ and then follow P from $y_1 = z_{i+1}$ to $z_m = w$.] Since a $v - w$ walk exists in G_1, then by a theorem there is a $v - w$ path in G, proving the claim.

We have shown that G_1 is a connected graph with exactly one cycle, namely C. Now pick any edge e' of C, and define $G_2 = G_1 - e'$; note that $e' \neq e$, since e is not an edge of G_1. Then G_2 has no cycles, since G_2 no longer has cycle C, which was the only cycle of G_1. In addition, G_2 is connected, by exactly the same argument as in the above claim.

Thus, G_2 is a connected, acyclic graph, i.e., G_2 is a tree.
By a theorem, then, we have $|E(G_2)| = |V(G_2)| - 1$.

However, we have $E(G_2) = E(G) \setminus \{e, e'\}$, so that $|E(G_2)| = |E(G)| - 2$.
We also have $V(G_2) = V(G_1) = V(G)$. Therefore,

$$|E(G)| = |E(G_2)| + 2 = |V(G_2)| - 1 + 2 = |V(G)| + 1$$ QED