Math 271, Sections 01,02, Spring 2019

Solutions to Selected Problems from PS#6–10

Comments in [square brackets] are not part of the proof, but just occasional side explanations.

1.5, #5a: Determine whether \( S = \{(1, 1, 2, 0), (1, -1, 1, 1), (3, 1, 0, 0)\} \) is linearly independent in \( \mathbb{R}^4 \).

**Answer/Proof.** We wish to determine whether there exist \( a, b, c \in \mathbb{R} \) such that
\[
a(1, 1, 2, 0) + b(1, -1, 1, 1) + c(3, 1, 0, 0) = (0, 0, 0, 0)
\]
besides \( a = b = c = 0 \). The above equation is equivalent to saying
\[
a + b + 3c = 0, \quad a - b + c = 0, \quad 2a + b = 0, \quad \text{and} \quad b = 0.
\]
The last equation says \( b = 0 \), so the third equation says \( 2a = 0 \) and hence \( a = 0 \). Thus, the second equation says \( c = 0 \). So the only choice of \( a, b, c \in \mathbb{R} \) satisfying the original vector equation is \( a = b = c = 0 \).

This is precisely what it means to say that \( S \) is linearly independent. QED

[Note: We could have set up a matrix and do Gaussian elimination to solve the above system of four linear equations in three variables, but in this case we could just read things off.]

1.5, #5c: Determine whether \( S = \{x^2 + x + 1, -x^2 + 2x, x^2 + 2, x^2 - x\} \) is linearly independent in \( P_2(\mathbb{R}) \).

**Answer/Proof.** We wish to determine whether there exist \( a, b, c, d \in \mathbb{R} \) such that
\[
a(x^2 + x + 1) + b(-x^2 + 2x) + c(x^2 + 2) + d(x^2 - x) = 0,
\]
besides \( a = b = c = d = 0 \). The above equation is
\[
(a - b + c + d)x^2 + (a + 2b + 2c - d)x + (a + 2c) = 0,
\]
which is equivalent to saying
\[
a - b + c + d = 0, \quad a + 2b + 2c - d = 0, \quad \text{and} \quad a + 2c = 0.
\]
Since these are \( m = 3 \) homogeneous linear equations in \( n = 4 > m \) unknowns, Corollary 1.5.13 says that there is a nontrivial solution for \( a, b, c, d \).

This is precisely what it means to say that \( S \) is not linearly independent. QED

1.6, #2b: Find a basis for and dimension of the subspace of \( \mathbb{R}^5 \) defined by
\[
\begin{align*}
2x_1 - x_2 + 3x_3 &= 0 \\
x_1 + 4x_2 - x_5 &= 0 \\
x_1 + x_2 - x_3 + x_4 + x_5 &= 0 \\
2x_1 + 2x_2 + x_3 + (1/2)x_4 &= 0
\end{align*}
\]

**Answer/Proof.**
\[
\begin{bmatrix}
2 & -1 & 3 & 0 & 0 \\
1 & 4 & 0 & 0 & -1 \\
1 & 1 & -1 & 1 & 1 \\
2 & 2 & 1 & 1/2 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & 0 & 0 & -1 \\
-2R1 & 2 & -1 & 3 & 0 & 0 \\
-1R1 & 1 & 1 & -1 & 1 & 1 \\
-2R1 & 2 & 2 & 1 & 1/2 & 0 & 0
\end{bmatrix}
\]
This reduced echelon form shows has two free variables: \( x_4 \) and \( x_5 \). Thus, by Corollary 1.6.15, the solution set \( W \) has \( \dim(W) = 2 \), and the basis vectors are given by all ways of setting one free variable to 1 and the rest to 0.

Setting \( x_4 = 1, x_5 = 0 \) gives \( x_1 = -2/3, x_2 = 1/6, \) and \( x_3 = 1/2 \), yielding the vector \((-2/3, 1/6, 1/2, 1, 0)\).

Setting \( x_4 = 0, x_5 = 1 \) gives \( x_1 = -7/9, x_2 = 4/9, \) and \( x_3 = 2/3 \), yielding the vector \((-7/9, 4/9, 2/3, 0, 1)\).

Replacing by scalar multiples to avoid annoying denominators, then, \( \{(-4,1,3,6,0),(-7,4,6,0,9)\} \) is a basis for \( W \).

1.6, \#2d: Find a basis for and dimension of the subspace of \( P_3(\mathbb{R}) \) defined by 
\[ \{ p \in P_3(\mathbb{R}) \mid p(2) = p(-1) = 0 \}. \]

**Answer/Proof.** Call the set above \( W \). Given \( p \in P_3(\mathbb{R}) \), writing \( p(x) = a + bx + cx^2 + dx^3 \), we have \( p(-1) = a - b + c - d, \) and \( p(2) = a + 2b + 4c + 8d \). Setting both to 0, we must solve the system represented by
\[
\begin{bmatrix}
1 & 2 & 4 & 8 & 0 \\
1 & -1 & 1 & -1 & 0
\end{bmatrix},
\]
which after a quick row reduction [omitted here] gives
\[
\begin{bmatrix}
1 & 0 & 2 & 2 & 0 \\
0 & 1 & 1 & 3 & 0
\end{bmatrix}.
\]
The free variables are \( c,d \), with solution \( a = -2c - 2d \) and \( b = -c - 3d \), for \( c,d \in \mathbb{R} \). Thus, by Corollary 1.6.15, \( \dim(W) = 2 \). Setting \( c = 1, d = 0 \) gives \( a = -2 \) and \( b = -1 \), yielding the polynomial \( x^2 - x - 2 \). Setting \( c = 0, d = 1 \) gives \( a = -2 \) and \( b = -3 \), yielding the polynomial \( x^3 - 3x - 2 \). Thus, \( S = \{x^2 - x - 2, x^3 - 3x - 2\} \) is a basis for \( W \).

[Note: an alternate way to finish, after having done the row reduction and solved for \( a,b \) in terms of the free variables, is to say:

\[ W = \{(-2c - 2d) + (-c - 3d)x + cx^2 + dx^3 \mid c,d \in \mathbb{R}\} = \text{Span}\{-2 - x + x^2, -2 - 3x + x^3\}. \]

We know this set \( S = \{x^2 - x - 2, x^3 - 3x - 2\} \) must be linearly independent by argument on page 54, so \( S \) is a basis for \( W \).]

1.6, \#3: Prove Corollary 1.6.14: Let \( W \) be a subspace of a finite-dimensional vector space \( V \). Then \( \dim(W) \leq \dim(V) \). Furthermore, \( \dim(W) = \dim(V) \) if and only if \( W = V \).

**Proof.** For the first statement: Let \( n = \dim(V) < \infty \). By definition of dimension, \( V \) has a basis \( S \) with exactly \( n \) elements.

Let \( T \subseteq W \) be a basis for \( W \). [I believe I said, but perhaps never wrote, that you may assume \( W \) has a basis, i.e., that you may assume such a set \( T \) exists.] Then \( T \) is a linearly independent subset of \( V \),
while $S$ is a subset of $V$ with $n$ elements and spanning $V$. By Theorem 1.6.10, then, $T$ cannot have more than $n$ elements. That is, $\dim(W) = \#T \leq n = \dim(V)$. \hspace{1cm} \text{QED (first statement)}

For the second statement, we do an if-and-only-if proof:

$(\Rightarrow)$: We already know $W \subseteq V$, so it remains to show that $V \subseteq W$. Since $\dim(W) = \dim(V) = n < \infty$, we have bases $S$ and $T$ for $V$ and $W$, respectively, each with $n$ elements.

Suppose there exists some $\vec{x} \in V$ with $\vec{x} \notin W$. Then $\vec{x} \notin \text{Span}(T)$. Since $T$ is linearly independent, Lemma 1.6.8 shows that $T \cup \{\vec{x}\}$ is also linearly independent. However, $T \cup \{\vec{x}\}$ has $n+1 > n$ elements, contradicting Theorem 1.6.10, which says that no linearly independent subset of $V$ can have more than $n$ elements. Contradiction! Thus, no such $\vec{x}$ can exist. That is, every $\vec{x} \in V$ must also belong to $W$.

\hspace{1cm} \text{QED (\Rightarrow)}.

$(\Leftarrow)$: Since $W = V$, we obviously have $\dim(W) = \dim(V)$.

\hspace{1cm} \text{QED}

[Side note: It’s actually not too hard to prove the thing I said you could assume, that $W$ has a basis. Here’s a way to do that, mimicking the proof of Theorem 1.6.6.

Let $S_0 = \emptyset$, which is linearly independent. If $S_0$ spans $W$, then it’s a basis, and we’re done. Otherwise, there exists $\vec{w}_1 \in W \setminus \text{Span}(S_0)$.

By Lemma 1.6.8, the set $S_1 = S_0 \cup \{\vec{w}_1\} \subseteq W$ is linearly independent. If $S_1$ spans $W$, then it’s a basis, and we’re done. Otherwise, there exists $\vec{w}_2 \in W \setminus \text{Span}(S_2)$.

By Lemma 1.6.8, the set $S_2 = S_1 \cup \{\vec{w}_2\} \subseteq W$ is linearly independent. Continuing in this fashion, we keep constructing bigger and bigger linearly independent sets $S_i \subseteq W$, stopping only if $\text{Span}(S_i) = W$.

It suffices to show that this eventually happens.

If we get as far as constructing $S_{n+1}$, then we have a linearly independent set

$$S_{n+1} = \{y_1, \ldots, y_{n+1}\} \subseteq W \subseteq V$$

with $n+1 > n = \dim(V)$ elements. But that’s impossible by Theorem 1.6.10. So the process must have stopped before step $n+1$. That is, there is some $i \leq n$ such that $S_i$ is a basis for $W$. \hspace{1cm} \text{QED}

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1.6, #5(a): Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $V$, and let $\dim(W_1) = n_1$ and $\dim(W_2) = n_2$. Prove that $\dim(W_1 \cap W_2) \leq \min\{n_1, n_2\}$.

**Proof.** Without loss of generality, $n_1 \leq n_2$. So we must show $\dim(W_1 \cap W_2) \leq n_1$.

By an old theorem (Theorem 1.2.13), $W_1 \cap W_2$ is a subspace of $V$. It’s contained in $W_1$, so it is also a subspace of $W_1$. Thus, Corollary 1.6.14 (applied to $W_1 \cap W_2 \subseteq W_1$) says that $\dim(W_1 \cap W_2) \leq \dim(W_1) = n_1$.

\hspace{1cm} \text{QED}

1.6, #5(b): Show by examples that if $n_1 = n_2 = 2$ in 1.6 #5(a), all the values $\dim(W_1 \cap W_2) = 0, 1, 2$ are possible.

**Answer/Proof.** Example 1: $V = \mathbb{R}^4$, $W_1 = \text{Span}(\{\vec{e}_1, \vec{e}_2\})$, and $W_2 = \text{Span}(\{\vec{e}_3, \vec{e}_4\})$.

The set $\{\vec{e}_1, \vec{e}_2\}$ is linearly independent since it is has two elements, neither of which is a scalar multiple of the other. [Or, because it is a subset of the standard basis, and any subset of a linearly independent set is still linearly independent.] So $\dim(W_1) = 2$. Similarly, $\dim(W_2) = 2$.

We have $W_1 = \{(a, b, 0, 0) | a, b \in \mathbb{R}\}$, and $W_2 = \{(0, 0, c, d) | c, d \in \mathbb{R}\}$.

So $W_1 \cap W_2 = \{(0, 0, 0, 0)\}$ is the trivial subspace, with dimension 0, as desired.

Example 2: $V = \mathbb{R}^3$, $W_1 = \text{Span}(\{\vec{e}_1, \vec{e}_2\})$, and $W_2 = \text{Span}(\{\vec{e}_2, \vec{e}_3\})$.

As in Example 1, $\dim(W_1) = \dim(W_2) = 2$.

We have $W_1 = \{(a, b, 0) | a, b \in \mathbb{R}\}$, and $W_2 = \{(0, b, c) | b, c \in \mathbb{R}\}$.

So $W_1 \cap W_2 = \{(0, b, 0) | b \in \mathbb{R}\}$, which has dimension 1, as desired, because the spanning set $\{\vec{e}_2\}$ has one element and is linearly independent, since that one element is nonzero.
Example 3: $V = W_1 = W_2 = \mathbb{R}^2$. Then $\dim(W_1) = \dim(W_2) = \dim(\mathbb{R}^2) = 2$. And $W_1 \cap W_2 = \mathbb{R}^2$ has dimension 2, as desired.

1.6, #7(b): Find a basis of $\mathbb{R}^4$ containing the linearly independent set $S = \{(0, 0, 2, 0, -1), (-1, 0, 0, 1, 0)\}$.

**Answer (Sketch).** Since $S$ has exactly two elements, neither of which is a scalar multiple of the other, $S$ is linearly independent.

Let $T$ be the standard basis $\{e_1, \ldots, e_5\}$ for $\mathbb{R}^5$.

Check, by trying to solve, that $e_1 \notin \text{Span}(S)$, and so let $S_1 = S \cup \{e_1\}$. Then $S_1$ is linearly independent, by Lemma 1.6.8.

Check, by trying to solve, that $e_2 \notin \text{Span}(S_1)$, and so let $S_2 = S_1 \cup \{e_2\} = S \cup \{e_1, e_2\}$. Then $S_2$ is linearly independent, by Lemma 1.6.8.

Check, by trying to solve, that $e_3 \notin \text{Span}(S_2)$, and so let $S_3 = S_2 \cup \{e_3\} = S \cup \{e_1, e_2, e_3\}$. Then $S_3$ is linearly independent, by Lemma 1.6.8.

Stop. $S_3$ is a set of five linearly independent vectors. Therefore $W = \text{Span}(S_3)$ is a subspace of $\mathbb{R}^5$ with $\dim(W) = 5 = \dim(\mathbb{R}^5)$. By Corollary 1.6.14, we have $W = \mathbb{R}^5$. That is, $S_3$ is a linearly independent set spanning $\mathbb{R}^5$, i.e., it is a basis for $\mathbb{R}^5$. QED

1.6, #15(a): Find a basis for and dimension of $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2} \left( \mathbb{R} \right) \mid a + 2b = c - 3d = 0 \right\}$.

**Answer/Proof.** The system of equations

\[
\begin{align*}
    a + 2b &= 0 \\
    c - 3d &= 0
\end{align*}
\]

is already in reduced echelon form, with free variables $b$ and $d$. So $\dim(W) = 2$, the number of free variables. Setting $b = 1$, $d = 0$ gives $a = -2$ and $c = 0$, yielding $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$. Setting $b = 0$, $d = 1$ gives $a = 0$ and $c = 3$, yielding $\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$. So $\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right\}$ is a basis for $W$. QED

Chapter 1 end, #3: Find a basis for and the dimension of $W = \{ p \in P_4(\mathbb{R}) \mid p(1) = p(-1) = 0 \}$.

**Solution.** Write $p \in P_4(\mathbb{R})$ as $p(x) = a + bx + cx^2 + dx^3 + ex^4$. Then $p \in W$ if and only if $a + b + c + d + e = 0$ and $a - b + c - d + e = 0$. Row reducing gives

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
\end{bmatrix} \rightarrow \cdots \rightarrow 
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

[Computational details omitted.] There are three free variables (namely, $c, d, e$), so $\dim(W) = 3$. Setting $c = 1$ and $d = e = 0$ gives $a = -1$, $b = 0$, i.e., $x^2 - 1$. The next basis vector (from $d = 1$) is $x^3 - x$, and the third is $x^4 - 1$. So $\{x^2 - 1, x^3 - x, x^4 - 1\}$ is a basis for $W$.

2.1, #3: Determine whether $T : V \to W$ is linear in each case:

(a): $V = \mathbb{R}^2$, $W = \mathbb{R}^4$, and $T((a_1, a_2)) = (a_2, a_1, a_1, a_2)$. **Yes, Linear.**

**Respects +:** Given $\vec{x}, \vec{y} \in V$, write $\vec{x} = (a_1, a_2)$ and $\vec{y} = (b_1, b_2)$. Then $T(\vec{x} + \vec{y}) = T((a_1 + b_1, a_2 + b_2)) = (a_2 + b_2, a_1 + b_1, a_1 + b_2 + b_2) = (a_2, a_1, a_1) + (b_2, b_1, b_1, b_2) = T(\vec{x}) + T(\vec{y})$.

**Respects s.m.:** Given $\vec{x} \in V$ and $c \in \mathbb{R}$, write $\vec{x} = (a_1, a_2)$. Then $T(c \vec{x}) = T((ca_1, ca_2)) = (ca_2, ca_1, ca_1, ca_2) = c(a_2, a_1, a_1, a_2) = cT(\vec{x})$ QED

(b): $V = \mathbb{R}^3$, $W = \mathbb{R}^3$, and $T((a_1, a_2, a_3)) = (a_1 + 2a_2, a_3 - a_2)$. **No, Not Linear.** We have $T(\vec{0}) = (1, 0, 0) \neq \vec{0}$, which we saw is impossible for a linear map.

(g): $V = \mathbb{R}$, $W = (0, \infty)$ with the operations of Example (1.1.4c), and $T(x) = e^x$. **Yes, Linear.**
Respects +: Given \( x, y \in V \), we have
\[
T(x + y) = e^{x+y} = e^x \cdot e^y = T(x) + T(y)
\] [since the + operation on \( W \) is usual multiplication]

Respects s.m.: Given \( x \in V \) and \( c \in \mathbb{R} \), we have
\[
T(cx) = e^{cx} = (e^c)^x = c \cdot T(x)
\] [since the s.m. operation on \( W \) is usual exponentiation] QED

2.1, #5(a): Let \( V = C^\infty(\mathbb{R}) \) and \( D : V \to V \) by \( D(f) = f' \). Prove that \( D \) is linear.

Proof. Given \( f, g \in V \), we have \( D((f + g)) = (f + g)' = f' + g' = D(f) + D(g) \).

Given \( f \in V \) and \( c \in \mathbb{R} \), we have \( D(cf) = (cf)' = c f' = c D(f) \) QED

2.1, #11: Let \( V = \mathbb{R}^2 \) and \( W = P_3(\mathbb{R}) \). If \( T : V \to W \) is linear with \( T((1,1)) = x + x^2 \) and \( T((3,0)) = x - x^3 \), what is \( T((2,2)) \)?

Solution. Since \( T \) is linear, \( T((2,2)) = T(2(1,1)) = 2T((1,1)) = 2(x + x^2) = 2x + 2x^2 \).

[Yup, it turned out that the value of \( T((3,0)) \) was irrelevant.]

2.2, #3(a): Let \( V = \mathbb{R}^3, W = \mathbb{R}^4 \), and \( T : V \to W \) by \( T(\vec{x}) = (x_1 - x_2, x_2 - x_3, x_1 + x_2 - x_3, x_3 - x_1) \).
Find the matrix of \( T \) with respect to the standard bases on \( V \) and \( W \).

Solution. \[
T(\vec{e}_1) = T((1,0,0)) = (1,0,1,-1) = 1\vec{e}_1 + 0\vec{e}_2 + 1\vec{e}_3 - 1\vec{e}_4.
\]
\[
T(\vec{e}_2) = T((0,1,0)) = (-1,1,1,0) = -1\vec{e}_1 + 1\vec{e}_2 + \vec{e}_3 + 0\vec{e}_4.
\]
\[
T(\vec{e}_3) = T((0,0,1)) = (0,-1,-1,1) = 0\vec{e}_1 - 1\vec{e}_2 - 1\vec{e}_3 + 1\vec{e}_4.
\]

So [putting these 4-tuples of coefficients down columns], the matrix is
\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
1 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix}
\]

2.2, #3(b): Let \( V = \mathbb{R}^n, W = \mathbb{R}^1, \vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \), and \( T : V \to W \) by \( T(\vec{v}) = a_1 v_1 + \cdots + a_n v_n \).
Find the matrix of \( T \) with respect to the standard bases on \( V \) and \( W \).

Solution. We use \( \alpha = \{\vec{e}_1, \ldots, \vec{e}_n\} \) as basis for \( V \), and \( \beta = \{1\} \) as basis for \( W \).
For each \( i = 1, \ldots, n \), we have \( T(\vec{e}_i) = a_i = a_i \cdot 1 \).
So [putting these \( n \) single numbers down columns], the matrix is \( [T]_\alpha^\beta = [a_1 \ a_2 \ \cdots \ a_n] \).

2.2, #6(a): Let \( V \) have basis \( \alpha = \{\vec{v}_1, \ldots, \vec{v}_n\} \), and let \( c \in \mathbb{R} \). Compute \([cI]_\alpha^\alpha\).

Solution. \( cI(\vec{v}_i) = c\vec{v}_i = c\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n \), and similarly \( cI(\vec{v}_i) = c\vec{v}_i \) for each \( i = 1, \ldots, n \).
So [putting these \( n \)-tuples of coefficients down columns], the matrix is
\[
[cI]_\alpha^\alpha = \begin{bmatrix}
c & 0 & 0 & \cdots & 0 \\
0 & c & 0 & \cdots & 0 \\
0 & 0 & c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{bmatrix}
\]

[That is, it is an \( n \times n \) diagonal matrix, with each diagonal entry being \( c \).]

2.2, #6(b): Let \( V \) have basis \( \alpha = \{\vec{v}_1, \ldots, \vec{v}_n\} \), and let \( k < n \). Define \( T : V \to V \) by \( T(a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n) = a_1 \vec{v}_1 + \cdots + a_k \vec{v}_k \).
Compute \([T]_\alpha^\alpha\).

Solution. We have \( T(\vec{v}_i) = \vec{v}_i \), and in fact \( T(\vec{v}_i) = \vec{v}_i \) for each \( i = 1, \ldots, k \). But \( T(\vec{v}_i) = \vec{0} \) for \( i = k+1, \ldots, n \). So [putting the \( n \)-tuples of coefficients down columns], the matrix is
2.2, #9(a): Let \( \alpha = \{(1, 2), (3, -4)\} \) and \( \beta = \{\vec{e}_1, \vec{e}_2\} \) (bases for \( \mathbb{R}^2 \)). Compute \([I]_\alpha^\beta\).

**Solution.** Write \( \vec{v}_1 = (1, 2) \) and \( \vec{v}_2 = (3, -4) \). We compute
\[
I \vec{v}_1 = (1, 2) = 1\vec{e}_1 + 2\vec{e}_2, \quad \text{and} \quad I \vec{v}_2 = (3, -4) = 3\vec{e}_1 - 4\vec{e}_2.
\]
So [putting the pairs of coefficients down columns], the matrix is \([I]_\alpha^\beta = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}\).

2.2, #12(a): Let \( V = M_{m \times n}(\mathbb{R}) \) and \( W = M_{m \times n}(\mathbb{R}) \). Prove that the transpose mapping \( T : V \to W \) by \( T(A) = A^t \) is linear.

**Proof.** [Recall the following notation from the PS #9 handout: Given \( A \in V \), we can write \( A = [a_{ij}] \); that is, for each \( i, j \), let \( a_{ij} \) denote the \((i, j)\)-th entry of \( A \). In that notation, we have \( A^t = [a_{ji}] \), since the \((i, j)\)-th entry of \( A^t \) is precisely \( a_{ji} \).]

Given \( A, B \in V \), write \( A = [a_{ij}] \) and \( B = [b_{ij}] \). Then
\[
T(A + B) = T([a_{ij}] + [b_{ij}]) = T([a_{ij} + b_{ij}]) = [a_{ij} + b_{ij}] = [a_{ji}] + [b_{ji}] = T(A) + T(B).
\]
In addition for the same \( A \), and given \( c \in \mathbb{R} \), we have
\[
T(cA) = T(c[a_{ij}]) = T([ca_{ij}]) = [ca_{ij}] = c[a_{ji}] = cT(A).
\]
**QED**

**PS #9 non-book #1(b):** Let \( \alpha = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\} \), which is a basis for \( \mathbb{R}^3 \).

For \( (x_1, x_2, x_3) \in \mathbb{R}^3 \), compute \([(x_1, x_2, x_3)]_\alpha\).

**Solution.** We need to solve \( (x_1, x_2, x_3) = y_1(1, 0, 0) + y_2(1, 1, 0) + y_3(1, 1, 1) \). Row reduction gives
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
x_1 \xrightarrow{\cdots} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
x_1 - x_2
\]
[computation skipped here]. So \( y_1 = x_1 - x_2, y_2 = x_2 - x_3, \) and \( y_3 = x_3 \).

That is, \([(x_1, x_2, x_3)]_\alpha = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix} \]

**PS #9 non-book #2:** Let \( T : P_3(\mathbb{R}) \to \mathbb{R}^4 \) by \( T(p) = (p(0), p'(0), p''(0), p'''(0)) \), which is linear. Compute the matrix of \( T \) with respect to the standard bases on \( P_3(\mathbb{R}) \) and \( \mathbb{R}^4 \).

**Solution.** Since \( p_1 = 1 \) has \( p_1' = 0 \), we have \( T(1) = (1, 0, 0, 0) \).

Since \( p_2 = x \) has \( p_2' = 1 \) and \( p_2'' = 0 \), we have \( T(x) = (0, 1, 0, 0) \).

Since \( p_3 = x^2 \) has \( p_3' = 2x \) and \( p_3'' = 2 \), we have \( T(x^2) = (0, 0, 2, 0) \).

Since \( p_4 = x^3 \) has \( p_4' = 3x^2 \), \( p_4'' = 6x \), and \( p_4''' = 6 \), we have \( T(x^3) = (0, 0, 0, 6) \).

So [putting these in columns] the matrix is \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 6
\end{bmatrix}
\]
PS #9 non-book #3: Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(p) = xp(x) - p'(x)$. Let $\alpha = \{1, x, x^2\}$ and
$\beta = \{1, x, x^2, x^3\}$ be the standard bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$. (a) Prove that $T$ is linear. (b) Compute $[T]_\alpha^\beta$. (c) Compute $[2x^2 - x - 3]_\alpha$ and $[T(2x^2 - x - 3)]_\beta$. (d) Verify $[T(2x^2 - x - 3)]_\beta = [T]_\alpha^\beta[2x^2 - x - 3]_\alpha$.

Solution. (a) (Respects +): Given $p, q \in P_2(\mathbb{R})$, we have

$$T(p + q) = x((p + q)(x)) - (p + q)'(x) = xp(x) + xq(x) - p'(x) - q'(x) = xp(x) - p'(x) + xq(x) - q'(x) = T(p) + T(q)$$

(Respects s.m.): Given $p \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$, we have

$$T(cp) = x((cp)(x)) - (cp)'(x) = c(xp(x) - p'(x)) = cT(p).$$

(b) $T(1) = x(1) - 0 = x = 0 + 1x + 0x^2 + 0x^3.$

Thus, the (standard) basis

$$\{x, x^2, x^3\},$$

which is also a linearly independent set (having one nonzero element), is a basis for $\text{Ker}(T)$.

So $[T]_\alpha^\beta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (c) $2x^2 - x - 3 = -3(1) - 1(x) + 2(x^2)$, so $[2x^2 - x - 3]_\alpha = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$.

$$T(2x^2 - x - 3) = x(2x^2 - x - 3) - (4x - 1) = 2x^3 - x^2 - 7x + 1,$$ so $[T(2x^2 - x - 3)]_\beta = \begin{bmatrix} 1 \\ -7 \\ -1 \\ 2 \end{bmatrix}$.

(d) $[T]_\alpha^\beta[2x^2 - x - 3]_\alpha = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 - 4 \\ -1 \\ 2 \end{bmatrix} = [T(2x^2 - x - 3)]_\beta$, as desired.

2.3, #1(f): Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ by differentiation. Find a basis for the domain $P_n(\mathbb{R})$ so that the first $\dim(\text{Ker}(T))$ vectors are a basis for $\text{Ker}(T)$.

Solution. Solving $T(p) = 0$ gives $p' = 0$, so $p(x) = c$ is constant. And conversely, if $p$ is constant, then $p' = 0$. Thus,

$$\text{Ker}(T) = \{c \mid c \in \mathbb{R}\} = \text{Span}\{1\}.$$ So $\{1\}$, which is also a linearly independent set (having one nonzero element), is a basis for $\text{Ker}(T)$. Thus, the (standard) basis $\{1, x, x^2, \ldots, x^n\}$ for $P_n(\mathbb{R})$ fits the desired conditions.

2.3, #3(c): Let $T : \mathbb{R}^6 \to \mathbb{R}^3$ be multiplication by $\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 2 & 0 \end{bmatrix}$. Find bases for $\text{Ker}(T)$ and $\text{Im}(T)$.

Solution. Call the original matrix $A$. Row reduction on $A$ leads to the echelon form

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix},$$
or if you prefer reduced echelon form,

$$C = \begin{bmatrix} 1 & 0 & 1 & 0/2 & 1/2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & -1/2 \end{bmatrix}.$$

[Details of the computation omitted here.] Since there are pivots in columns 1, 2, 4, we pull the corresponding columns from $A$ to deduce that

$$\{(1, -1, 0), (0, 1, 1), (-1, 1, 2)\}$$
is a basis for $\text{Im}(T)$. 


[Side note: since \( \text{dim}(\text{Im}(T)) = 3 = \text{dim}(\mathbb{R}^3) \), in fact the map \( T \) is onto, and the image is all of \( \mathbb{R}^3 \). So actually, any basis for \( \mathbb{R}^3 \), such as the standard basis, is a basis for \( \text{Im}(T) = \mathbb{R}^3 \). But I’m writing up this solution to exemplify how to do such a problem in general; and in general, you need to pull the pivot columns from \( A \).

As for the kernel, calling the input variables \( x_1, \ldots, x_6 \), either echelon form shows that \( x_3, x_5, x_6 \) are free, with

\[
x_1 = -x_3 - \frac{1}{2}x_5 - \frac{1}{2}x_6, \quad x_2 = -3x_3 - x_5 - x_6, \quad x_4 = -\frac{1}{2}x_5 + \frac{1}{2}x_6.
\]

[The reduced echelon form \( C \) gives this immediately; the non-reduced form \( B \) gives this after a little extra work.] So

\[
\{(-1, -3, 1, 0, 0, 0), \left( -\frac{1}{2}, -1, 0, -\frac{1}{2}, 1, 0 \right), \left( -\frac{1}{2}, -1, 0, 1, 0, 1 \right) \}
\]

is a basis for \( \text{Ker}(T) \).

[Or, to simplify a bit by multiplying by some nonzero scalars to clear denominators, we could also use \{(-1, -3, 1, 0, 0, 0), (-1, -2, 0, -1, 2, 0), (-1, -2, 0, 1, 0, 2) \} as a basis for \( \text{Ker}(T) \).]

---

2.3, #7(a): Let \( \vec{v} = (1, 1) \) and \( \vec{w} = (2, 1) \). Construct a linear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \vec{v} \in \text{Ker}(T) \) and \( \vec{w} \in \text{Im}(T) \).

**Solution.** Writing the matrix for \( T \) as \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), the condition that \( \vec{v} \in \text{Ker}(T) \) says that \( A\vec{v} = \vec{0} \), i.e., that \( a + b = 0 \) and \( c + d = 0 \). Meanwhile, to ensure that \( \vec{w} \in \text{Im}(T) \), we just need to pick some element \( \vec{x} \) of \( \mathbb{R}^2 \) and require that \( T\vec{x} = \vec{w} \). In particular, we can require that \( T((1, 0)) = \vec{w} \), which is to say that \( (a, c) = \vec{w} \).

So choosing \( (a, c) = (2, 1) \) and requiring \( a + b = c + d = 0 \) induces us to declare \( T \) to be given by

\[
T(\vec{x}) = A\vec{x}, \text{ where } A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.
\]

Sure enough, \( T\vec{v} = (0, 0) \) and \( T\vec{e}_1 = \vec{w} \), so that \( \vec{v} \in \text{Ker}(T) \) and \( \vec{w} \in \text{Im}(T) \).

[Note: there are other correct answers, obtained by choosing different vectors \( \vec{x} \) to satisfy \( T\vec{x} = \vec{w} \) (in place of \( \vec{w} = \vec{e}_1 \) as we used above). However, it turns out that any such choice will result in a nonzero scalar multiple of the matrix \( A \) above.]

---

2.3, #13: Let \( V = M_{n \times n}(\mathbb{R}) \), and let \( T : V \to V \) by \( T(A) = (1/2)(A + A^t) \). (a) Prove that \( \text{Ker}(T) \) is the subspace of skew-symmetric matrices in \( V \), and \( \text{Im}(T) \) is the subspace of symmetric matrices in \( V \). (b) Compute \( \text{dim}(\text{Im}(T)) \). (c) Use Rank-Nullity and part (b) to compute \( \text{dim}(\text{Im}(T)) \).

**Solution.**

(a) Let \( X \subseteq V \) be the set of skew-symmetric matrices in \( V \). We need to show \( X = \text{Ker}(T) \):

\( \subseteq \): Given \( A \in X \), we have \( A^t = -A \) by definition of skew-symmetric. Thus,

\[
T(A) = \frac{1}{2}(A + A^t) = \frac{1}{2}(A - A) = \frac{1}{2}O = O,
\]

where \( O \in V \) denotes the zero-matrix. Thus, \( A \in \text{Ker}(T) \).

\( \supseteq \): Given \( A \in \text{Ker}(T) \), we have \( T(A) = O \), so \( \frac{1}{2}(A + A^t) = O \). Multiplying both sides by 2, we have \( A + A^t = O \), and hence \( A^t = -A \). Thus, \( A \in X \). \( \text{QED Ker} \)

Let \( Y \subseteq V \) be the set of symmetric matrices in \( V \). We need to show \( Y = \text{Im}(T) \):

\( \subseteq \): Given \( B \in Y \), we have \( B^t = B \) by definition of symmetric. [We now need to think of some \( A \in V \) such that \( T(A) = B \). Fortunately, after some doodling of examples on scratch paper, it occurs to us that choosing \( A = B \) itself works!]

Note that:

\[
T(B) = \frac{1}{2}(B + B^t) = \frac{1}{2}(B + B) = \frac{1}{2}(2B) = B.
\]

Thus, \( B \in \text{Im}(T) \), since it is \( T \) of some element of \( V \). (Namely, itself!)

\( \supseteq \): Given \( B \in \text{Im}(T) \), there exists \( A \in V \) such that \( T(A) = B \). That is, \( B = \frac{1}{2}(A + A^t) \). Thus, recalling that the map \( M \to M^t \) is linear (2.2 Exercise 12(a), earlier in this handout), we have
\[ B^t = \left[ \frac{1}{2} (A + A^t) \right]^t = \frac{1}{2} (A^t + (A^t)^t) = \frac{1}{2} (A^t + A) = \frac{1}{2} (A + A^t) = B. \]

That is, \( B \) is symmetric, and hence \( B \in Y \).

QED Im

[Note 1: Some students find it easier or more conceptual to do these proof with specific reference to the entries of \( A, B \in V \). So for example, to prove that \( Y \supseteq \text{Im}(T) \), we could phrase the proof as follows: Given \( B \in \text{Im}(T) \), there exists \( A \in V \) such that \( T(A) = B \). Write \( A = [a_{ij}] \) and \( B = [b_{ij}] \). Then because \( B = \frac{1}{2} (A + A^t) \), we have \( b_{ij} = \frac{1}{2} (a_{ij} + a_{ji}) \) for all \( i, j \). Hence,
\[
b_{ji} = \frac{1}{2} (a_{ji} + a_{ij}) = \frac{1}{2} (a_{ij} + a_{ji}) = b_{ij} \text{ for all } i, j.
\]

That is, \( B^t = B \), and hence \( B \in Y \).

QED (≥)

Note 2: To prove \( Y \subseteq \text{Im}(T) \), given \( B \in Y \), we chose \( A = B \) to make \( T(A) = B \) happen. But there are a lot of choices that will also work; the choice of \( A = B \) is merely the simplest.

For example, if we write a given \( B \in Y \) as \( B = [b_{ij}] \), then we could define \( A = [a_{ij}] \in V \) by
\[
a_{ij} = \begin{cases} 
2b_{ij} & \text{if } i < j, \\
b_{ij} & \text{if } i = j, \\
0 & \text{if } i > j.
\end{cases}
\]

(That is, the entries of \( A \) below the diagonal are all 0; the ones on the diagonal match those from \( B \); and the ones above the diagonal are double those from \( B \).) Then using the fact that \( b_{ij} = b_{ji} \), it is fairly quick to show that \( \frac{1}{2} (a_{ij} + a_{ji}) = b_{ij} \) for all \( i, j \), and hence \( T(A) = B \).]

(b) [Note: to compute \( \text{dim}(\text{Ker}(T)) = \text{dim}(X) \), we will find a basis for \( X \) and count its elements. To do that, note that an arbitrary skew-symmetric matrix has zeros down the diagonal, arbitrary real numbers in all the positions above the diagonal, and the negatives of those real numbers below the diagonal. That thinking motivates what I’m about to do here.]

Each position above the diagonal in an \( n \times n \) matrix has (row,column) position \( (i, j) \) with \( 1 \leq i < j \leq n \). For each such pair \( i, j \), define a matrix \( M_{ij} \) by:

- The \( (i, j) \)-entry of \( M_{ij} \) is 1,
- The \( (j, i) \)-entry of \( M_{ij} \) is -1, and
- All other entries of \( M_{ij} \) are 0.

[For example, if \( n = 3 \), then we are talking about three matrices, since there are three ways to choose integers \( i, j \) with \( 1 \leq i < j \leq 3 \). And
\[
M_{12} = \begin{bmatrix} 0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, \quad M_{13} = \begin{bmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 \end{bmatrix}
\]

are the resulting matrices.]

Let \( S = \{ M_{ij} | 1 \leq i < j \leq n \} \). We claim that \( S \) is a basis for \( X \).

To see that \( S \) is linearly independent, given scalars \( a_{ij} \) (for \( 1 \leq i < j \leq n \)) with \( \sum_{i < j} a_{ij} M_{ij} = O \), the matrix on the right has \( ij \)-entry \( a_{ij} \) for each such \( i, j \). Since this entry must be 0 in the zero matrix \( O \), we have \( a_{ij} = 0 \), for each such \( i, j \), as desired.

To see that \( \text{Span}(S) = X \), given \( A \in X \), write \( A = [a_{ij}] \). Since \( A \in X \), we have \( A^t = -A \), and hence \( a_{ji} = -a_{ij} \) for each \( i, j \). For \( i = j \), this equation gives \( 2a_{ii} = 0 \), and therefore \( a_{ii} = 0 \). Thus, \( A = \sum_{i < j} a_{ij} M_{ij} \in \text{Span}(S) \), proving the claim.

Therefore, \( \text{dim}(\text{Ker}(T)) = \text{dim}(X) = \#S = \#\{(i, j) | 1 \leq i < j \leq n\} \). Hence, \( j \) can range from 2 to \( n \), and for each such \( j \), \( i \) can be be any of the \( j - 1 \) integers from 1 to \( j - 1 \). Thus,
\[
\text{dim}(\text{Ker}(T)) = \sum_{j=2}^{n} (j - 1) = 1 + 2 + \cdots + (n - 1) = \frac{n(n - 1)}{2}
\]
So finally, then \( \text{(b): Using part (a), we compute} \)
\[
\text{That is, }\frac{1}{2}(x_1 - x_2 + x_3)\]
\[
\text{Dimensions of the row reduction omitted here.} \]
\[
\text{That is, the dimension of the kernel of } T \text{ is simply the number of entries in an } n \times n \text{ matrix that lie strictly above the diagonal.} \]

(c) By Rank-Nullity,
\[
\dim(\operatorname{Im}(T)) = \dim(V) - \dim(\operatorname{Ker}(T)) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.
\]

[Which, by the way, is the number of entries on or below the diagonal, in an \( n \times n \) matrix.]

**PS #10 non-book #1:** Let \( \alpha = \{(1,0,1), (0,1,1), (1,1,0)\} \) and \( \beta = \{(1,1), (-1,1)\} \) be ordered bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \), respectively. Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be a linear transformation whose matrix with respect to \( \alpha \) and \( \beta \) is
\[
[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 4 \\ -1 & -3 & 2 \end{bmatrix}.
\]

(a) Work out a formula for \( T((x_1, x_2, x_3)) \), and (b) Find the matrix of \( T \) with respect to the standard bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \).

**Solution.** (a): First, let’s find the coordinate vector \( [(x_1, x_2, x_3)]_\alpha \). That is, find the unique \( y_1, y_2, y_3 \in \mathbb{R} \) such that \( (x_1, x_2, x_3) = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3 \), where \( \alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \). So we solve
\[
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{2}(x_1 - x_2 + x_3) = \frac{1}{2}(-x_1 + x_2 + x_3) = \frac{1}{2}(x_1 + x_2 - x_3)
\]
[Details of the row reduction omitted here.] That is, \( [(x_1, x_2, x_3)]_\alpha = \frac{1}{2} \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix} \). Thus,
\[
[T(x_1, x_2, x_3)]_\beta = [T]_\alpha^\beta [(x_1, x_2, x_3)]_\alpha = \begin{bmatrix} 1 & 0 & 4 \\ -1 & -3 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5x_1 + 3x_2 - 3x_3 \\ 4x_1 - 6x_3 \end{bmatrix}
\]

Finally, then [since \( [\vec{w}]_\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \)] means that \( \vec{w} = y_1(1, 1) + y_2(-1, 1) \), we have
\[
T(x_1, x_2, x_3) = \left( \frac{5}{2}x_1 + \frac{3}{2}x_2 - \frac{3}{2}x_3 \right)(1, 1) + (2x_1 - 3x_3)(-1, 1) = \left( \frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{3}{2}x_3, \frac{9}{2}x_1 + \frac{3}{2}x_2 - \frac{9}{2}x_3 \right)
\]

(b): Using part (a), we compute
\[
T(\vec{e}_1) = \left( \frac{1}{2}, \frac{9}{2} \right) = \frac{1}{2} \vec{e}_1 + \frac{9}{2} \vec{e}_2, \quad T(\vec{e}_2) = \left( \frac{3}{2}, \frac{3}{2} \right) = \frac{3}{2} \vec{e}_1 + \frac{3}{2} \vec{e}_2, \quad T(\vec{e}_3) = \left( \frac{3}{2}, -\frac{9}{2} \right) = \frac{3}{2} \vec{e}_1 - \frac{9}{2} \vec{e}_2.
\]
So \( [T]_{\text{std}} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{9}{2} & \frac{3}{2} & -\frac{9}{2} \end{bmatrix} \).