1.1, #3 [properties 4,6]: Prove vector space axioms 4 and 6 for $\mathbb{R}^n$.

**Solution:** Axiom 4: ∀ $\vec{x} \in \mathbb{R}^n$, $\exists -\vec{x} \in \mathbb{R}^n$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$.]

**Proof:** Given $\vec{x} \in \mathbb{R}^n$, write $\vec{x} = (x_1, \ldots, x_n)$. Let $-\vec{x} = (-x_1, \ldots, -x_n) \in \mathbb{R}^n$. Then $\vec{x} + (-\vec{x}) = (x_1 + (-x_1), \ldots, x_n + (-x_n)) = (0, \ldots, 0) = \vec{0}$. QED

Axiom 6: ∀ $\vec{x} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, $(c + d)\vec{x} = c\vec{x} + d\vec{x}$.

**Proof:** Given $\vec{x} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, write $\vec{x} = (x_1, \ldots, x_n)$. Then

$(c + d)\vec{x} = (c + d)(x_1, \ldots, x_n) = ((c + d)x_1, \ldots, (c + d)x_n) = (cx_1 + dx_1, \ldots, cx_n + dx_n)
= (cx_1, \ldots, cx_n) + (dx_1, \ldots, dx_n) = c(x_1, \ldots, x_n) + d(x_1, \ldots, x_n) = c\vec{x} + d\vec{x}$. QED

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1.1, #4 [properties 3,8]: Prove vector space axioms 3 and 8 for $P_n(\mathbb{R})$.

**Solution:** Axiom 3: $\exists \vec{0} \in P_n(\mathbb{R})$ s.t. $\forall f \in P_n(\mathbb{R})$, $f + \vec{0} = f$.

**Proof:** Let $\vec{0} = 0 + 0x + \cdots + 0x^n \in P_n(\mathbb{R})$. Given $f \in P_n(\mathbb{R})$, write $f = a_0 + \cdots + a_n x^n$. Then

$f + \vec{0} = \sum_{i=0}^n a_i x^i + \sum_{i=0}^n 0x^i = \sum_{i=0}^n (a_i + 0)x^i = \sum_{i=0}^n a_i x^i = f$. QED

Axiom 8: $\forall f \in P_n(\mathbb{R})$, $1f = f$.

**Proof:** Given $f \in P_n(\mathbb{R})$, write $f = a_0 + \cdots + a_n x^n$. Then

$1f = 1 \sum_{i=0}^n a_i x^i = \sum_{i=0}^n (1a_i)x^i = \sum_{i=0}^n a_i x^i = f$. QED

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1.1, #6c: Say which vector space axioms fail to hold for $\mathbb{R}^2$ with addition by $(x_1, x_2) +' (y_1 + y_2) = (0, x_1 + y_2)$ and with usual scalar multiplication. [And for those that fail, prove that they fail.]

**Solution:** Axiom 1 (Associativity of +') fails. [To prove this, we must find one choice of $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ such that $(\vec{x} + ' \vec{y}) + ' \vec{z} \neq \vec{x} + ' ((\vec{y} + ' \vec{z})].$]

**Proof of 1's failure:** Let $\vec{x} = (1, 0)$ and $\vec{y} = \vec{z} = (0, 0)$. Then

$(\vec{x} + ' \vec{y}) + ' \vec{z} = ((1, 0) +' (0, 0)) +' (0, 0) = (1, 1) +' (0, 0) = (0, 0)
\neq (0, 1) = (0, 1) +' (0, 0) = (0, 1) +' ((0, 0) +' (0, 0)) = \vec{x} +' (\vec{y} +' \vec{z})$. QED [1 fails]

Axiom 2 (Commutativity of +') fails. [To prove this, we must find one choice of $\vec{x}, \vec{y} \in \mathbb{R}^2$ such that $\vec{x} +' \vec{y} \neq \vec{y} +' \vec{x}$.]

**Proof of 2's failure:** Let $\vec{x} = (1, 0)$ and $\vec{y} = (0, 0)$. Then

$\vec{x} +' \vec{y} = (1, 0) +' (0, 0) = (1, 0) \neq (0, 0) = (0, 0) +' (1, 0) = \vec{y} +' \vec{x}$. QED [2 fails]

Axiom 3 (Identity of +' fails. [To prove this, we must show that for all $\vec{z} \in \mathbb{R}^2$ (that could potentially be an additive identity), there is some $\vec{x} \in \mathbb{R}^2$ so that $\vec{x} +' \vec{z} \neq \vec{x}$].

**Proof of 3's failure:** Given $\vec{z} \in \mathbb{R}^2$, write $\vec{z} = (z_1, z_2)$. Let $\vec{x} = (1, 0)$. Then

$\vec{x} +' \vec{z} = (1, 0) +' (z_1, z_2) = (0, 1 + z_2) \neq (1, 0) = \vec{x}$. QED [3 fails]

Axiom 4 (Inverses of +') fails, for the simple reason that the statement of the axiom involves the zero element, which we just showed doesn’t exist.

Axiom 5 (Distributivity 1) holds [no proof required, but I suggest you make sure you can do the proof].

Axiom 6 (Distributivity 2) fails. [To prove this, we must find one choice of $\vec{x} \in \mathbb{R}^2$ and of $c, d \in \mathbb{R}$ such that $(c + d)\vec{x} \neq c\vec{x} +' d\vec{x}]$

**Proof of 6's failure:** Let $\vec{x} = (1, 0) \in \mathbb{R}^2$, let $c = 1 \in \mathbb{R}$, and let $d = 0 \in \mathbb{R}$. Then
\((c + d)\vec{x} = 1(1, 0) = (1, 0) \neq (0, 1) = (1, 0) + (0, 0) = c\vec{x} + d\vec{x}\). 

Axiom 7 (Associativity of scalar mult) holds. [But no proof required.]

Axiom 8 (Identity of scalar mult) holds. [But no proof required.]

[Note: I made a lot of arbitrary choices in the proofs above that may make you think, “Where did that come from?” But that is what a proof of a statement like “there exists \(x\) such that” looks like, or what a proof of the failure of a statement like “for all \(x\)” looks like. You just need one example of such a thing. Even if there are many different choices that will work, but you have to pick a specific one. I’d recommend choosing simple examples, involving simple numbers (like 0 and 1), simple functions (like constant functions), etc., as you mess around on scratch paper to find something that works.]

1.1, #7c: Say which vector space axioms fail to hold for \(F(\mathbb{R})\) with addition by \(f + g = f \circ g\) and with usual scalar multiplication. [And for those that fail, prove that they fail.]

Solution: Axiom 1 (Associativity of \(\circ\)) holds. [But no proof required.]

Axiom 2 (Commutativity of \(\circ\)) fails. [To prove this, we must find one choice of \(f, g \in F(\mathbb{R})\) such that \(f + g \neq g + f\).]

Proof of 2's failure: Let \(f = 0 \in F(\mathbb{R})\) and \(g = 1 \in F(\mathbb{R})\) [the constant-zero and constant-one functions]. Then \(f + g = f \circ g = 0 \neq 1 = g \circ f = g + f\). QED [2 fails]

Axiom 3 (Identity of \(\circ\)) holds, with \(\vec{0} = i\), where \(i \in F(\mathbb{R})\) is the function \(i(x) = x\). [No proof required, but make sure you see why.]

Axiom 4 (Inverses of \(\circ\)) fails. [To prove this, we must prove \(\exists f \in F(\mathbb{R})\) s.t. \(\forall g \in F(\mathbb{R}), f + g \neq 0\).

That is, there is (at least) one \(f\) such that for any possible candidate \(g\) to play the role of \(-f\), actually \(g\) doesn’t work as \(-f\). And remember, “zero” here is the function \(i(x) \equiv x\), not the usual zero.]

Proof of 4's failure: Let \(f = 0 \in F(\mathbb{R})\).

Given \(g \in F(\mathbb{R})\), we have \(f + g = f \circ g = 0 \neq i = \vec{0}\) QED [4 fails]

[FYI: the reason \(0 \neq i\) is that these are both functions, and they differ in value at, say, \(x = 1\), where \(0(1) = 0\) but \(i(1) = 1 \neq 0\). In particular, they are different not because their formulas are different, but rather because there is at least one point in the domain where they have different values.]

Axiom 5 (Distributivity 1) fails. [To prove this, we must find one choice of \(f, g \in F(\mathbb{R})\) and of \(c \in \mathbb{R}\) such that \(c(f + g) \neq cf + cg\).]

Proof of 5's failure: Let \(f = g = i \in F(\mathbb{R})\) [that is, \(f(x) = x\) and \(g(x) = x\) for all \(x \in \mathbb{R}\)], and let \(c = 2 \in \mathbb{R}\). Then for any \(x \in \mathbb{R}\),

\[c(f + g)(x) = 2f(g(x)) = 2f(x) = 2x,\]

and \((cf + cg)(x) = 2f(2g(x)) = 2f(2x) = 2(2x) = 4x.\)

So at \(x = 1\) (say), we get \(c(f + g)(1) = 2 \neq 4 = (cf + cg)(1)\), so \(c(f + g) \neq cf + cg\). QED [5 fails]

Axiom 6 (Distributivity 2) fails. [To prove this, we must find one choice of \(f \in F(\mathbb{R})\) and of \(c, d \in \mathbb{R}\) such that \((c + d)f \neq cf + df\].

Proof of 6's failure: Let \(f = 1 \in F(\mathbb{R})\), and let \(c = d = 1 \in \mathbb{R}\). Then \((c + d)f) = 2f = 2 \neq 1 = f \circ (cf) \circ (df) = cf + df\). QED [6 fails]

Axiom 7 (Associativity of scalar mult) holds. [But no proof required.]

Axiom 8 (Identity of scalar mult) holds. [But no proof required.]

1.1, #8a: Let \(V\) be a vector space, and \(\vec{x}, \vec{y}, \vec{z} \in V\) such that \(\vec{x} + \vec{y} = \vec{x} + \vec{z}\). Prove that \(\vec{y} = \vec{z}\).

Proof: Given \(\vec{x}, \vec{y}, \vec{z} \in V\) as above, we have

\[\vec{y} = \vec{y} + \vec{0} = \vec{0} + \vec{y} = (\vec{x} + \vec{z}) + \vec{y} = (-\vec{x} + \vec{x}) + \vec{y} = -\vec{x} + (\vec{x} + \vec{y}) = -\vec{x} + (\vec{x} + \vec{z}) + \vec{z} = (\vec{x} + \vec{z}) + \vec{z} + \vec{0} = \vec{z},\]

where the first and last equalities are by axiom #3, the second and tenth by #2, the third and ninth by #4, fourth and eighth by #2, fifth and seventh by #1, and sixth by hypothesis. QED
To prove the claim, we do the following computation:
\[(a + b)(\vec{x} + \vec{y}) = a\vec{x} + b\vec{x} + a\vec{y} + b\vec{y},\]

where the first equality is by axiom #5, the second by #6, and the third (the fact that we can ignore parentheses) by #1. QED

1.1, #8b: Let \( V \) be a vector space, and let \( \vec{x}, \vec{y} \in V \) and \( a, b \in \mathbb{R} \). Prove that \( (a + b)(\vec{x} + \vec{y}) = a\vec{x} + b\vec{x} + a\vec{y} + b\vec{y} \).

**Proof.** Given \( \vec{x}, \vec{y} \in V \) and \( a, b \in \mathbb{R} \), we have
\[(a + b)(\vec{x} + \vec{y}) = (a + b)\vec{x} + (a + b)\vec{y} = (a\vec{x} + b\vec{x}) + (a\vec{y} + b\vec{y}) = a\vec{x} + b\vec{x} + a\vec{y} + b\vec{y},\]
where the first equality is by axiom #5, the second by #6, and the third (the fact that we can ignore parentheses) by #1. QED

1.2, #1: Let \( V \) be a vector space. Prove that \( \{\vec{0}\} \subseteq V \) is a subspace.

**Proof.** Let \( W = \{\vec{0}\} \).

(Nonempty): Since \( \vec{0} \in W \), we have \( W \neq \emptyset \).

(Closed under +): Given \( \vec{x}, \vec{y} \in W \), we have \( \vec{x} = \vec{y} = \vec{0} \), so \( \vec{x} + \vec{y} = \vec{0} + \vec{0} = \vec{0} \in W \). Thus, \( W \) is closed under addition.

(Closed under sc mult): Given \( \vec{x} \in W \) and \( c \in \mathbb{R} \), we have \( \vec{x} = \vec{0} \). We claim that \( c\vec{0} = \vec{0} \).

To prove the claim, we do the following computation:
\[c\vec{0} = c\vec{0} + c\vec{0} + (c\vec{0} + -c\vec{0}) = (c\vec{0} + c\vec{0}) + -c\vec{0} = c(\vec{0} + \vec{0}) + -c\vec{0} = c\vec{0} + -c\vec{0} = \vec{0},\]
where the first and fifth equalities are by axiom #3, the second and last by #4, the third by #1, and the fourth by #5. So \( c\vec{0} = \vec{0} \in W \). Thus, \( W \) is closed under scalar multiplication. QED

[Note: The proof of the claim may seem hard to dream up. But this is a strategy we saw in class (first thought of by some genius long ago): when trying to prove that \( \vec{0} \), it might help to start by proving that \( \vec{0} = \vec{0} \). So for the claim above, after I couldn’t think of a simpler idea, I started on scratch paper from \( c\vec{0} + c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} \), and then I added the negative of \( c\vec{0} \) to both sides.]

1.2, #2b Let \( V_1 = \{f \in F(\mathbb{R}) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\} \). Prove that \( V_1 \) is a subspace of \( F(\mathbb{R}) \).

**Proof.** (Nonempty): Let \( f(x) = 0 \in F(\mathbb{R}) \). Then \( f(x) = 0 = f(-x) \) for all \( x \in \mathbb{R} \), so \( f \in V_1 \). Thus, \( V_1 \neq \emptyset \).

(Closed under +): Given \( f, g \in V_1 \), then for any \( x \in \mathbb{R} \), we have
\[(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x).\]
Thus, \( f + g \in V_1 \).

(Closed under sc mult): Given \( f \in V_1 \) and \( c \in \mathbb{R} \), then for any \( x \in \mathbb{R} \), we have
\[(cf)(x) = c(f(x)) = c(f(-x)) = (cf)(-x).\]
Thus, \( cf \in V_1 \). QED

1.2, #3d: Let \( V = \mathbb{R}^3 \), and \( W = \{(a_1, a_2, a_3) \in V \mid a_1, a_2, a_3 \geq 0\} \). Is \( W \) a subspace of \( V \)?

**Answer/Proof.** No. \( W \) is not closed under scalar multiplication. Let \( \vec{w} = (1, 1, 1) \in W \). Then
\[-1\vec{w} = (-1, -1, -1) \notin W.\]

1.2, #3b: Let \( V = P_n(\mathbb{R}) \), and \( W = \{p \in V \mid p(\sqrt{2}) = 0\} \). Is \( W \) a subspace of \( V \)?

**Answer/Proof.** Yes. (Nonempty): The zero polynomial \( \vec{0} \in P_n(\mathbb{R}) \) has \( \vec{0}(\sqrt{2}) = 0 \), so \( \vec{0} \in W \).

(Closed under +): Given \( p, q \in P_n(\mathbb{R}) \), we have
\[(p + q)(\sqrt{2}) = p(\sqrt{2}) + q(\sqrt{2}) = 0 + 0 = 0, \text{ so } p + q \in W.\]

(Closed under sc mult): Given \( p \in P_n(\mathbb{R}) \) and \( c \in \mathbb{R} \), we have
\((cp)(\sqrt{2}) = cp(\sqrt{2}) = c(0) = 0\), so \(cp \in W\). QED

1.2, #3i: Let \(V = P_n(\mathbb{R})\), and \(W = \{p \in V \mid p(1) = 1 \text{ and } p(2) = 0\}\). Is \(W\) a subspace of \(V\)?

**Answer/Proof.** No. Any subspace must contain the zero vector, but the zero vector of \(V\) is the zero polynomial \(\vec{0} \in P_n(\mathbb{R})\). Since \(\vec{0}(1) = 0 \neq 1\), we have \(\vec{0} \notin W\). So \(W\) is not a subspace. QED

1.2, #5: Let \(V\) be a vector space, \(W \subseteq V\) a subspace, and \(\vec{y} \in V\). Let \(\vec{y} + W = \{\vec{x} \in V \mid \exists \vec{w} \in W \text{ s.t. } \vec{x} = \vec{y} + \vec{w}\}\). Prove that \(\vec{y} + W\) is a subspace of \(V\) if and only if \(\vec{y} \in W\).

**Proof (\(\Rightarrow\)).** Since \(\vec{y} + W\) is a subspace, we have \(\vec{0} \in \vec{y} + W\). Thus, there exists \(\vec{w} \in W\) such that \(\vec{0} = \vec{y} + \vec{w}\). [Remember, the statement “\(\vec{w} \in \vec{y} + W\)” means that there exists \(\vec{w} \in W\) such that \(\vec{v} = \vec{y} + \vec{w}\).] Hence, \(\vec{y} = -\vec{w} = (-1)\vec{w} \in W\) as desired, because \(W\) is closed under scalar multiplication.

(\(\Leftarrow\)) (Nonempty): Since \(\vec{0} \in W\), we have \(\vec{y} + \vec{0} \in \vec{y} + W\). So \(\vec{y} + W \neq \emptyset\).

**Note:** I didn’t bother simplifying \(\vec{y} + \vec{0}\) to \(\vec{y}\), because I’m focused solely on the goal. Here, the goal is to show \(\vec{y} + W\) has something in it. I don’t need to simplify that something; I only need to show that something exists. Of course, it doesn’t hurt to do the simplification, but that only makes the solution longer, for no benefit.

**Closed under +:** Given \(\vec{x}_1, \vec{x}_2 \in \vec{y} + W\), there exist \(\vec{w}_1, \vec{w}_2 \in W\) such that \(\vec{x}_1 = \vec{y} + \vec{w}_1\) and \(\vec{x}_2 = \vec{y} + \vec{w}_2\). Then \(\vec{x}_1 + \vec{x}_2 = (\vec{y} + \vec{w}_1) + (\vec{y} + \vec{w}_2) = \vec{y} + (\vec{w}_1 + \vec{w}_2) \in \vec{y} + W\), since \(\vec{w}_1 + \vec{w}_2 \in W\), which is in turn because \(\vec{y}, \vec{w}_1, \vec{w}_2 \in W\), and \(W\) is closed under +.

**Closed under sc mult:** Given \(\vec{x} \in \vec{y} + W\) and \(c \in \mathbb{R}\), there exists \(\vec{w} \in W\) such that \(\vec{x} = \vec{y} + \vec{w}\). Then \(c\vec{x} = c(\vec{y} + \vec{w}) = c\vec{y} + cw \in \vec{y} + (c - 1)\vec{y} + cw \in \vec{y} + W\), since \((c - 1)\vec{y} + cw \in W\), which is in turn because \(\vec{y}, \vec{w} \in W\), and \(W\) is closed under + and sc mult. QED

**Note:** How did I think of that? Well, it’s an if-and-only-if proof, so one chapter is (\(\Rightarrow\)) and the other is (\(\Leftarrow\)). To prove (\(\Rightarrow\)), I structure that chapter of the proof around my goal, that \(\vec{y} \in W\). So I know that “\(\vec{y} \in W\)” has to be the last line of the chapter. And then I think about what aspect of the hypothesis (that \(\vec{y} + W\) is a subspace) might help me get there. Similarly, for proving (\(\Leftarrow\)), my goal is to show that \(\vec{y} + W\) is a subspace, so I set up the standard proof for that. Then any time I’m stuck as I go through those three steps, I check to see if using the hypothesis (that \(\vec{y} \in W\) might help me.)

1.3, #3: Let \(S = \{1, 1 + x, 1 + x + x^2\} \subseteq P_2(\mathbb{R})\). Show that \(\text{Span}(S) = P_2(\mathbb{R})\).

**Proof.** (\(\subseteq\)): Given \(f \in \text{Span}(S)\), \(f\) is a linear combination of the elements of \(S \subseteq P_2(\mathbb{R})\). Since \(P_2(\mathbb{R})\) is closed under the operations, we have \(f \in P_2(\mathbb{R})\).

(\(\supseteq\)): Given \(f \in P_2(\mathbb{R})\), write \(f = a + bx + cx^2\). Then \(f = (a - b)(1) + (b - c)(1 + x) + c(1 + x + x^2) \in \text{Span}(S)\) QED

1.3, #7: Let \(V\) be a vector space, \(W \subseteq V\) a subspace, and \(S \subseteq W\) a subset. Prove that \(\text{Span}(S) \subseteq W\).

**Proof.** Given \(\vec{x} \in \text{Span}(S)\), there exist an integer \(n \geq 0\) and \(\vec{v}_1, \ldots, \vec{v}_n \in S\) and \(a_1, \ldots, a_n \in \mathbb{R}\) such that \(\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n\). Since each \(\vec{v}_i \in S \subseteq W\), we have \(a_i\vec{v}_i \in W\) by closure under scalar multiplication. Thus, by closure under addition, \(\vec{x} \in W\).

1.4, #1a: Determine whether \(S = \{(1, 1), (1, 3), (0, 2)\} \subseteq \mathbb{R}^2\) is linearly dependent or independent.

**Answer/Proof.** Note \(1(1, 1) + (-1)(1, 3) + 1(0, 2) = (1 - 1 + 0, 1 - 3 + 2) = (0, 0)\).

So \(S\) is linearly dependent. QED

1.4, #1e: Determine whether \(S = \{x^2 + 1, x - 7\} \subseteq P_2(\mathbb{R})\) is linearly dependent or independent.

**Answer/Proof.** Given \(a, b \in \mathbb{R}\) such that \(a(x^2 + 1) + b(x - 7) = 0\), we have \(ax^2 + bx + (a - 7b) = 0\). Equating coefficients on both sides, \(a = b = a - 7b = 0\). In particular, \(a = b = 0\).
So $S$ is linearly independent.

1.4, #1h: Determine whether $S = \{e^x, \cos(x)\} \subseteq C(\mathbb{R})$ is linearly dependent or independent.

**Answer/Proof.** Given $a, b \in \mathbb{R}$ such that $ae^x + b\cos(x) = 0$, evaluating both sides at $x = \pi/2$ gives $ae^{\pi/2} + 0 = 0$, so $a = 0$. Thus, evaluating both sides of the original equation at $x = 0$ gives $0e^0 + b\cos(0) = 0$, so $b = 0$ also. Hence, $S$ is linearly independent. QED

1.4, #1j: Determine whether $S = \{\sin^2(x), \cos^2(x), -3\} \subseteq C(\mathbb{R})$ is linearly dependent or independent.

**Answer/Proof.** Note $3\sin^2(x) + 3\cos^2(x) + 1(-3) = 3 - 3 = 0$. So $S$ is linearly dependent. QED

1.4, #5: Let $\vec{v}, \vec{w} \in V$. Show that $\{\vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{v} + \vec{w}, \vec{v} - \vec{w}\}$ is linearly independent.

**Proof.** For convenience, let $S = \{\vec{v}, \vec{w}\}$ and $S' = \{\vec{v} + \vec{w}, \vec{v} - \vec{w}\}$.

$(\Rightarrow)$: Given $a, b \in \mathbb{R}$ such that $a(\vec{v} + \vec{w}) + b(\vec{v} - \vec{w}) = \vec{0}$, we have

$$(a + b)\vec{v} + (a - b)\vec{w} = a(\vec{v} + \vec{w}) + b(\vec{v} - \vec{w}) = \vec{0}.$$ 

Thus, because $S$ is linearly independent, we have $a + b = a - b = 0$. Therefore, $2a = (a + b) + (a - b) = 0 + 0 = 0$, so $a = 0$. Hence $0 + b = 0$, so $b = 0$ also. QED $(\Rightarrow)$

$(\Leftarrow)$: Given $a, b \in \mathbb{R}$ such that $a\vec{v} + b\vec{w} = \vec{0}$, we have

$$a + b \vec{v} + a - b \vec{w} = (a + b)\vec{v} + (a - b)\vec{w} = a\vec{v} + b\vec{w} = \vec{0}.$$ 

Thus, because $S'$ is linearly independent, we have $a + b = a - b = 0$. Therefore, $a = a + b = 0$ and $b = a - b = 0$. Hence, $\vec{v} = \vec{0}$ and $\vec{w} = \vec{0}$. QED

1.4, #8: Let $W_1$ and $W_2$ be subspaces of a vector space such that $W_1 \cap W_2 = \{\vec{0}\}$. Let $S_1 \subseteq W_1$ and $S_2 \subseteq W_2$ be linearly independent. Show that $S_1 \cup S_2$ is linearly independent.

**Proof.** Given distinct vectors $\vec{x}_1, \ldots, \vec{x}_n \in S_1 \cup S_2$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1\vec{x}_1 + \cdots + a_n\vec{x}_n = \vec{0}$,

we may assume WLOG that $\vec{x}_1, \ldots, \vec{x}_k \in S_1$ and $\vec{x}_{k+1}, \ldots, \vec{x}_n \in S_2$ for some $k$ between 0 and $n$. Define $\vec{v}$ to be the vector

$$\vec{v} = a_1\vec{x}_1 + \cdots + a_k\vec{x}_k = -a_{k+1}\vec{x}_{k+1} - \cdots - a_n\vec{x}_n.$$ 

Then since $\vec{v} = a_1\vec{x}_1 + \cdots + a_k\vec{x}_k$, we have $\vec{v} \in W_1$. In addition, since $\vec{v} = -a_{k+1}\vec{x}_{k+1} - \cdots - a_n\vec{x}_n$, we have $\vec{v} \in W_2$. That is, $\vec{v} \in W_1 \cap W_2$, and hence $\vec{v} = \vec{0}$.

Since $a_1\vec{x}_1 + \cdots + a_k\vec{x}_k = \vec{v} = \vec{0}$, we must have $a_1 = \cdots = a_k = 0$. (Since $S_1$ is linearly independent.) Similarly, since $a_{k+1}\vec{x}_{k+1} + \cdots + a_n\vec{x}_n = -\vec{v} = \vec{0}$, we also have $a_{k+1} = \cdots = a_n = 0$.

Thus, we have shown $a_i = 0$ for all $i = 1, \ldots, n$. QED

1.4, #9a: Let $V$ be a vector space and $S \subseteq V$. Suppose that $\text{Span}(S) = V$. Prove that for all $\vec{v} \in V$, $\{\vec{v}\} \cup S$ is linearly independent.

**Note:** This problem is actually incorrect as stated. There should be an extra restriction, that $\vec{v} \not\in S$. Otherwise, if $\vec{v}$ were in fact an element of $S$ itself, then $\{\vec{v}\} \cup S = S$. If in addition $S$ were linearly independent, then of course $\{\vec{v}\} \cup S = S$ would also be linearly independent. So the problem really should have asked us to prove this for all $\vec{v} \in V \setminus S$, not for all $\vec{v} \in V$.

**Proof.** Given $\vec{v} \in V$, we have $\vec{v} \in \text{Span}(S)$, so there exist $\vec{x}_1, \ldots, \vec{x}_n \in S$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $\vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$. Thus, $a_1\vec{x}_1 + \cdots + a_n\vec{x}_n + (-1)\vec{v} = \vec{v} - \vec{v} = \vec{0}$, which is a linear combination of elements of $\{\vec{v}\} \cup S$ with at least one coefficient (the $-1$) being nonzero, but adding up to the zero vector. Hence, $\{\vec{v}\} \cup S$ is linearly dependent. QED
Note: The above proof is acceptable for the purposes of homework, but as hinted above, something goes wrong if \( \vec{v} \in S \). What goes wrong? Well, in checking linear dependence, we really need the vectors in the linear combination to be distinct. So here’s a proof without that gap in it.]

Corrected Proof. Given \( \vec{v} \in V \setminus S \), we have \( \vec{v} \in V = \text{Span}(S) \), so there exist \( \vec{x}_1, \ldots, \vec{x}_n \in S \) and \( a_1, \ldots, a_n \in \mathbb{R} \) such that \( \vec{v} = a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n \).

Without loss of generality, we may assume that \( \vec{x}_1, \ldots, \vec{x}_n \) are all distinct. After all, if \( \vec{x}_i = \vec{x}_j \) for some \( i < j \), we may combine the two terms \( a_i \vec{x}_i \) and \( a_j \vec{x}_j \) into simply \( (a_i + a_j) \vec{x}_i \).

In addition, since \( \vec{v} \notin S \), we have \( \vec{v} \neq \vec{x}_i \) for each \( i \). Thus, \( a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n + (-1)\vec{v} = \vec{v} - \vec{v} = \vec{0} \), which is a linear combination of distinct elements of \( \{\vec{v}\} \cup S \), with at least one coefficient (the \( -1 \)) being nonzero, but adding up to the zero vector. Hence, \( \{\vec{v}\} \cup S \) is linearly dependent.

QED

1.5, #1c: Solve the following system of equations:

\[
\begin{align*}
2x_1 + 3x_2 - 3x_3 &= 0 \\
3x_1 + x_3 &= 0
\end{align*}
\]

Solution.\[
\begin{align*}
-R2 \begin{bmatrix} 2 & 3 & -3 & 0 \\ 3 & 0 & 1 & 0 \end{bmatrix} &\rightarrow (\!-\!1) \begin{bmatrix} -1 & 3 & -4 & 0 \\ 3 & 0 & 1 & 0 \end{bmatrix} \rightarrow -3R1 \begin{bmatrix} 1 & -3 & 4 & 0 \\ 3 & 0 & 1 & 0 \end{bmatrix} \\
&\rightarrow \frac{1}{3} \begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{9} & 0 \end{bmatrix}
\end{align*}
\]

Thus, \( x_3 \) is a free variable, with \( x_1 = -x_3/3 \) and \( x_2 = 11x_3/9 \). So the solution is

\[
\left\{ \begin{bmatrix} -\frac{t}{3} & \frac{11t}{9} & t \end{bmatrix} \mid t \in \mathbb{R} \right\}, \quad \text{or equivalently,} \quad \left\{ \begin{bmatrix} -3t & 11t & 9t \end{bmatrix} \mid t \in \mathbb{R} \right\}.
\]

1.5, #2d: Let \( V = P_3(\mathbb{R}) \), and let \( W = \{ p \in V \mid p(1) = p(3) = 0 \} \). Find a set of vectors spanning \( W \).

Solution. Given \( p \in P_3(\mathbb{R}) \), write \( p(x) = a + bx + cx^2 + dx^3 \). Then \( p \in W \) if and only if \( p(1) = p(3) = 0 \), i.e., if and only if

\[
a + b + c + d = a + 3b + 9c + 27d = 0.
\]

This is a system of two linear equations in four unknowns, which we solve now:

\[
\begin{align*}
-R1 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} &\rightarrow \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 8 & 26 \end{bmatrix} \rightarrow \\
-R2 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 13 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -3 & -12 \\ 0 & 1 & 4 & 13 \end{bmatrix}
\end{align*}
\]

Thus, the variables \( c \) and \( d \) are free, with \( a = 3c + 12d \) and \( b = -4c - 13d \). So

\[
W = \left\{ (3c + 12d) + (-4c - 13d)x + cx^2 + dx^3 \mid c, d \in \mathbb{R} \right\} = \left\{ c(3 - 4x + x^2) + d(12 - 13x + x^3) \mid c, d \in \mathbb{R} \right\}.
\]

That is, \( W = \text{Span}(S) \), where \( S = \{ 3 - 4x + x^2, 12 - 13x + x^3 \} \).

1.5, #3a: Let \( \vec{v} = (1, 1, 2) \) and \( S = \{(1, 1, 4), (-1, 1, 3), (0, 1, 0)\} \subseteq \mathbb{R}^3 \). Determine whether \( \vec{v} \in \text{Span}(S) \).

Answer/Proof. We wish to know whether there exist \( x, y, z \in \mathbb{R} \) such that \( x(1, 1, 4) + y(-1, 1, 3) + (0, 1, 0) = (1, 1, 2) \). In other words, does the following system have a solution?

\[
\begin{align*}
-R1 \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\rightarrow \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \\
-4R1 \begin{bmatrix} 4 & 3 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \\
&\rightarrow -\frac{1}{2}R3 \begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{5}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & \frac{4}{7} \end{bmatrix}
\end{align*}
\]

Thus, the system has the solution \( (x, y, z) = \left( \frac{5}{7}, -\frac{2}{7}, \frac{4}{7} \right) \). That is, the answer is yes, because \( \vec{v} = \frac{5}{7}(1, 1, 4) - \frac{2}{7}(-1, 1, 3) + \frac{4}{7}(0, 1, 0) \in \text{Span}(S) \).