Solutions to Midterm Exam 2

1. (10 points) For each of the following products of matrices, either compute the product or (briefly) explain why the product is not defined.

\[
\begin{align*}
1a. & \quad \begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & 0 \end{bmatrix} \\
1b. & \quad \begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\end{align*}
\]

\textbf{Solution.} (a) Not defined. Can't do $2 \times 3$ times $2 \times 3$, since $3 \neq 2$.

(b) \[= \begin{bmatrix} 3 - 6 & 2 + 1 & 6 - 1 \\ -1 + 0 & 0 - 4 & -2 + 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 5 \\ -1 & -4 & 2 \end{bmatrix}\]

2. (15 points) Let \(A = \begin{bmatrix} 3 & -1 & -5 \\ 1 & 0 & -2 \\ -3 & 1 & 3 \end{bmatrix}\). Compute \(A^{-1}\).

\textbf{Solution.} \[\frac{1}{3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 3 \\ -3 & 1 & 3 \end{bmatrix}\]

So \(A^{-1} = \begin{bmatrix} -3/2 & 3 & -1/2 \\ -1/2 & 0 & -1/2 \end{bmatrix}\), or, if you prefer, \(\frac{1}{2} \begin{bmatrix} -2 & 2 & -2 \\ -3 & 6 & -1 \end{bmatrix}\).

3. (15 points) Let \(A = \begin{bmatrix} 4 & -8 & 3 \\ -1 & 2 & 1 \\ 3 & -6 & 7 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\).

Take my word for it that \(B\) is an echelon form of \(A\).

(That is: \textbf{I'VE ALREADY DONE THE ROW REDUCTION}!)

3a. Find bases for the kernel \(\text{Ker}(A)\) and the image \(\text{Im}(A)\) of \(A\).

3b. Find the rank of \(A\) and the nullity of \(A\).

\textbf{Solution.} There are pivots in the first and third columns only; see \(B\).

(a) For \(\text{Ker}(A)\): Solving for pivot variables in terms of free variables:

The first row of the echelon form says \(x_1 = 2x_2 - 2x_4 + 3x_5\), and the second says \(x_3 = x_4 - 2x_5\). So

\[
\text{Ker}(A) = \begin{bmatrix} 2x_2 - 2x_4 + 3x_5 \\ x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} : x_2, x_4, x_5 \in \mathbb{R}
\]

Thus, \(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}\) is a basis for \(\text{Ker}(A)\).

For \(\text{Im}(A)\), we pull the pivot columns (first and third) of \(A\), so \(\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}\) is a basis for \(\text{Im}(A)\).

(b) There are two pivots, so \(\text{rank}(A) = 2\). There are three free variables, so \(\text{nullity}(A) = 3\).
4. (15 points) Let $\alpha = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$. Take my word for it that $\alpha$ is a basis for $\mathbb{R}^2$ and that $T$ is linear.

4a. Let $\beta$ be the standard basis for $\mathbb{R}^2$. Find the change-of-basis matrices $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$.

**Solution.** (a): The easy one is $Q = [I]_\alpha^\beta = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. So $[I]_\beta^\alpha = Q^{-1} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

(b): We have $[T]_\beta^\alpha = A$, so $[T]_\alpha^\beta = [I]_\beta^\alpha A[I]_\alpha^\beta = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$= \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}.$$

5. (15 points) Define $T : P_3(\mathbb{R}) \to \mathbb{R}^2$ by $(\alpha) = (p(-1), p''(2))$. You may take my word for it that $T$ is linear. Let $\alpha = \{1, x, x^2, x^3\}$, which is a basis for $P_3(\mathbb{R})$, and let $\beta = \{(1,0), (0,1)\}$, which is the standard basis for $\mathbb{R}^2$.

5a. Find the matrix $[T]_\alpha^\beta$.

5b. Is $T$ injective? Is $T$ surjective? Don’t forget to (briefly) justify your answers.

**Solution.** Since $1'' = x'' = 0$ and $(x^2)'' = 2$ and $(x^3)'' = 6x$, we compute

$$T(1) = (1,0), \ T(x) = (-1,0), \ T(x^2) = (1,2), \ T(x^3) = (-1,12).$$

So since each $(a,b)$ is $a\vec{e}_1 + b\vec{e}_2$, we get $[T]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 12 \end{bmatrix}$.

(b): The above matrix is already in (non-reduced) echelon form. It has two pivots. Since there is not a pivot in every column, the map $T$ is not injective. Since there is a pivot in every row, the map $T$ is surjective.

6. (15 points) Let $V,W$ be vector spaces, let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of $V$, and let $T : V \to W$ be a one-to-one linear map. Let $\beta = \{T\vec{v}_1, \ldots, T\vec{v}_n\}$. Prove that $\beta$ is linearly independent.

**Proof.** Given $c_1, \ldots, c_n \in \mathbb{R}$ such that $c_1T\vec{v}_1 + \cdots + c_nT\vec{v}_n = \vec{0}$, then $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \vec{0}$, since $T$ is linear. Therefore, since $T$ is one-to-one, we have $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$. Since $\alpha$ is linearly independent, it follows that $c_1 = \cdots = c_n = 0$. QED

7. (15 points) Recall that $\beta = \{B_1, B_2, B_3, B_4\}$ is a basis for $M_{2\times2}(\mathbb{R})$, where

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $C = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix}$. Define $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by $T(A) = AC$. Take my word for it that $T$ is linear. Find the matrix $[T]_\beta^\alpha$ representing $T$ with respect to the basis $\beta$.

**Solution.** We compute

$$T(B_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} = 4B_1 - 2B_2 + 0B_3 + 0B_4,$$
$$T(B_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = 1B_1 + 5B_2 + 0B_3 + 0B_4,$$
$$T(B_3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & -2 \end{bmatrix} = 0B_1 + 0B_2 + 4B_3 - 2B_4,$$
$$T(B_4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 1B_1 + 0B_2 + 0B_3 + 1B_4.$$
\[
T(B_4) = \begin{bmatrix}
0 & 0 \\
4 & -2 \\
0 & 1 \\
1 & 5
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
5 & 4
\end{bmatrix} = 0B_1 + 0B_2 + 1B_3 + 5B_4.
\]
So \( [T]_β^β = \begin{bmatrix}
4 & 1 & 0 & 0 \\
-2 & 5 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 0 & -2 & 5
\end{bmatrix} \).

**OPTIONAL BONUS.** (2 points) Let \( S, T : \mathbb{R}^9 \to \mathbb{R}^7 \) be linear maps. Recall that \( S + T \) is the linear map \( S + T : \mathbb{R}^9 \to \mathbb{R}^7 \) given by \((S + T)(\vec{x}) = S\vec{x} + T\vec{x}\).
Suppose that \( \text{rank}(S) = 6 \) and \( \text{rank}(T) = 2 \). Prove that \( \text{nullity}(S + T) \leq 5 \).

**Proof.** We claim that \( \text{Ker}(S + T) \subseteq S^{-1}(\text{Im}(T)) \). To prove the claim: given \( \vec{v} \in \text{Ker}(S + T) \), we have \( S\vec{v} + T\vec{v} = \vec{0} \). Thus, \( S\vec{v} = -T\vec{v} = T(-\vec{v}) \in \text{Im}(T) \). That is, \( \vec{v} \in S^{-1}(\text{Im}(T)) \), proving the claim.
Taking dimensions, then, we have \( \text{nullity}(S + T) = \dim(\text{Ker}(S + T)) \leq \dim\left(S^{-1}(\text{Im}(T))\right) \).
Recall Exercise 2.4.10(c), which says that if \( T : V \to W \) is linear and \( X \subseteq W \) is a subspace, we have \( \dim(T^{-1}(X)) = \dim(\text{Ker}(T)) + \dim(X) \).
Applying this fact to \( S \) with \( X = \text{Im}(T) \), we have \( \dim\left(S^{-1}(\text{Im}(T))\right) = \dim(\text{Ker}(S)) + \dim(\text{Im}(T)) \).
Combining this last equality with the inequality above, we have
\[
\text{nullity}(S + T) \leq \dim\left(S^{-1}(\text{Im}(T))\right) = \dim(\text{Ker}(S)) + \dim(\text{Im}(T))
\]
\[
= \text{nullity}(S) + \text{rank}(T) = \left( \dim(V) - \text{rank}(S) \right) + \text{rank}(T) = (9 - 6) + 2 = 5.
\]
QED

[Note: there are other ways to do this problem.]