1. (17 points) Find the set of all solutions \((x, y, z, w) \in \mathbb{R}^4\) to the following system of equations.

\[
egin{align*}
  x - 2y + 3z - 2w &= 2 \\
  2x + y + z + 6w &= 14 \\
  x + y + 4w &= 8
\end{align*}
\]

**Solution.** Row reduction gives

\[
\begin{align*}
  -2R_1 & \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 & 2 \\ 2 & 1 & 1 & 6 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 & 2 \\ 0 & 5 & -5 & 10 & 10 \end{bmatrix} \\
  -R_1 & \rightarrow \begin{bmatrix} 1 & 1 & 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & -3 & 6 & 6 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
  +2R_2 & \rightarrow \begin{bmatrix} 1 & -2 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -1 & 2 & 2 \end{bmatrix} \\
  -3R_2 & \rightarrow \begin{bmatrix} 0 & 3 & -3 & 6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

So \(z, w\) are the free variables (since there are pivots in the columns for \(x, y\)), and we have \(x = 6 - z - 2w\) and \(y = 2 + z - 2w\). So the solution set is

\[
\{(6 - z - 2w, 2 + z - 2w, z, w) \mid z, w \in \mathbb{R}\}
\]

2. (12 points) Let \(V\) be a vector space, and let \(S = \{\vec{x}_1, \ldots, \vec{x}_n\} \subseteq V\) be a finite subset. Define the following terms and phrases. You may use other standard terms without defining them.

2a. Span\((S)\).

2b. \(S\) is linearly independent.

**Answers.**

(a): Span\((S)\) is the set of all linear combinations of elements of \(S\).

(b): \(S\) is linearly independent if for all \(a_1, \ldots, a_n \in \mathbb{R}\) such that \(a_1\vec{x}_1 + \cdots + a_n\vec{x}_n = \vec{0}\), we have \(a_1 = \cdots = a_n = 0\).

3. (15 points) Prove the following theorem we have learned:

Let \(V\) be a vector space, and let \(W_1, W_2 \subseteq V\) be subspaces of \(V\). Then \(W_1 \cap W_2\) is a subspace of \(V\).

**Proof.** (Nonempty): We have \(\vec{0} \in W_1\) and \(\vec{0} \in W_2\), since \(W_1\) and \(W_2\) are subspaces of \(V\). Thus, \(\vec{0} \in W_1 \cap W_2\).

(Closure): Given \(\vec{x}, \vec{y} \in W_1 \cap W_2\) and \(c \in \mathbb{R}\), then since \(\vec{x}, \vec{y} \in W_1\), we have \(c\vec{x} + \vec{y} \in W_1\). Similarly, since \(\vec{x}, \vec{y} \in W_2\), we have \(c\vec{x} + \vec{y} \in W_2\). Thus, \(c\vec{x} + \vec{y} \in W_1 \cap W_2\). QED

4. (12 points) Working in the vector space \(\mathbb{R}^4\), let \(S = \{(2, 1, 0, 3), (0, 1, -1, 2)\}\). Is \((4, 3, 2, 1) \in \text{Span}(S)\)? Why or why not?

**Answer:** No. To see this:
Write \( \vec{x} = (2, 1, 0, 3) \) and \( \vec{y} = (0, 1, -1, 2) \), and \( \vec{v} = (4, 3, 2, 1) \). Suppose \( \vec{v} \in \text{Span}(S) \). Then there exist \( a, b \in \mathbb{R} \) such that \( \vec{v} = a\vec{x} + b\vec{y} \). That is,

\[
(4, 3, 2, 1) = a(2, 1, 0, 3) + b(0, 1, -1, 2),
\]
i.e.,

\[
(4, 3, 2, 1) = (2a, a + b, -b, 3a + 2b).
\]

Equating coefficients, we have

\[
2a = 4, \quad a + b = 3, \quad -b = 2, \quad 3a + 2b = 1.
\]

The first of these equations says \( a = 2 \), and the third says \( b = -2 \). But then \( a + b = 0 \), contradicting the second equation. Thus, no such \( a, b \) exist, so \( \vec{v} \not\in \text{Span}(S) \).

5. (15 points) Let \( V = F(\mathbb{R}) \), the vector space of functions from \( \mathbb{R} \) to \( \mathbb{R} \).

Let \( W = \{ f \in V \mid f(5) = 4f(2) \} \). Prove that \( W \) is a subspace of \( V \).

**Proof.** (Nonempty): Let \( g = 0 \), the zero-function. Then \( g(5) = 0 = 4 \cdot 0 = 4g(2) \), so \( g \in W \).

(Closure): Given \( f, g \in W \) and \( c \in \mathbb{R} \), we have

\[
(cf + g)(5) = c(f(5)) + g(5) = c(4f(2)) + 4g(2) = 4(cf(2) + g(2)) = 4((cf + g)(2)),
\]
and hence \( cf + g \in W \). QED

6. (17 points) Let \( V = P_2(\mathbb{R}) \), the vector space of polynomials of degree at most 2.

Let \( W = \{ p \in V \mid p(2) = 2p'(0) \} \).

**It is a fact, which you may assume, that** \( W \) **is a subspace of** \( P_2(\mathbb{R}) \).

Find a basis for \( W \).

**Solution.** Write \( p \in V \) as \( p(x) = ax + bx + cx^2 \), so \( p' = b + 2cx \). So the equation \( p(2) = 2p'(0) \) says \( a + 2b + 4c = 2b \), i.e., \( a = -4c \). So

\[
W = \{ (-4c) + bx + cx^2 \mid b, c \in \mathbb{R} \} = \{ bx + c(x^2 - 4) \mid b, c \in \mathbb{R} \} = \text{Span}(S),
\]

where \( S = \{ x, x^2 - 4 \} \). We have that \( S \) spans \( W \) by the above equality. Moreover, \( S \) is linearly independent because it has exactly two elements, and neither is a multiple of the other.

[An alternate way to see that \( S \) is linearly independent is to quote a Corollary that \( \dim(W) \) is the number of free variables, i.e., 2. Since \( S \) spans \( W \) and has the correct number of elements, the two-out-of-three Corollary then says that \( S \) is also linearly independent.]

Thus, \( S = \{ x, x^2 - 4 \} \) is a basis for \( W \).

7. (12 points) In this problem, \( a, b, c \in \mathbb{R} \) are three specific numbers that I am keeping secret.

7a. Consider the set \( S_1 = \left\{ \begin{bmatrix} 0 \\ 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 7 \\ c \end{bmatrix} \right\} \) of three vectors in \( \mathbb{R}^4 \).

Take my word for it that \( S_1 \) is linearly independent. Is \( \text{Span}(S_1) = \mathbb{R}^4 \)?
Answer “Yes,” “No,” or “Need more information.” Justify your answer.

7b. Consider the set 
\[ S_2 = \left\{ \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 6 \end{bmatrix} \right\} \] of four vectors in \( \mathbb{R}^4 \).

Take my word for it that \( \text{Span}(S_2) = \mathbb{R}^4 \). Is \( S_2 \) linearly independent?
Answer “Yes,” “No,” or “Need more information.” Justify your answer.

Solutions. (a): No. \( \mathbb{R}^4 \) has a basis with 4 elements, and hence a linearly independent set with 4 elements. By a Theorem, any spanning set must have at least as many elements as any linearly independent set; so any spanning set must have at least 4 elements. Therefore, since \( \#S_1 = 3 < 4 \), \( S_1 \) cannot span \( \mathbb{R}^4 \).

(b): Yes. Since \( S_2 \) spans \( \mathbb{R}^4 \) and has the same number of elements (namely, 4) as a basis, the two-out-of-three corollary says that \( S_2 \) is a basis for \( \mathbb{R}^4 \) and hence is linearly independent.

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**OPTIONAL BONUS.** (2 points) Let \( V \) be a vector space, let \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in V \), and let \( a_1, a_2, a_3, a_4 \in \mathbb{R} \). Suppose \( \dim(V) = 12 \), and let

\[ S = \{ a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 + a_4 \vec{v}_4, a_1 \vec{v}_1 + a_4 \vec{v}_3 + a_2 \vec{v}_4 \} \]

Suppose that the (five) elements of \( S \) are all distinct. Prove that the set \( S \) is linearly dependent.

**Proof.** Let \( T = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \} \), and let \( W = \text{Span}(T) \), which is a subspace of \( V \).
Since \( W \) is closed under the operations, it follows that \( S \subseteq W \). If \( S \) were linearly independent, then by a theorem from class and the book [Theorem 1.6.6], we have \( \#S \leq \#T \), since both are subsets of \( W \) with \( S \) being linearly independent and \( T \) spanning \( W \).
However, \( S \) has five elements, while \( T \) has only four. This is a contradiction. Thus, \( S \) cannot be linearly independent; it must be linearly dependent.

QED