What you need to know for Exam 2

The exam will cover Chapter 2. (Of course, you still need to know the material in Chapter 1, but the exam will not focus on those topics except as they arise in the newer material.) The following is a list of most of the topics covered. THIS IS NOT A COMPREHENSIVE LIST, BUT MERELY AN AID. As before, for any given concept or definition below, you should know the intuitive idea and the rigorous definition, and you should be able to use it correctly both in a proof and in a computational problem.

- **2.1:** Linear maps: definition and basic properties. The identity and zero maps. Proposition 2.1.14: a linear map \( T : V \to W \) is uniquely determined by its values on a given basis for \( V \).

- **2.2:** The matrix \([T]_{\alpha}^{\beta}\) (where \( T \) is linear, and \( V, W \) are finite-dimensional with bases \( \alpha \) and \( \beta \)). The \( n \times n \) identity matrix \( I_n \), or simply \( I \). Product of a matrix and a vector; Proposition 2.2.15. Multiplication-by-\( A \) (hereafter, \( T_A \)) is linear; Proposition 2.2.19.

- **2.3:** Kernel \( \ker(T) \) and image \( \text{im}(T) \) of \( T : V \to W \), and their basic properties. The nullity and rank of \( T \), defined by \( \text{nullity}(T) = \dim(\ker(T)) \) and \( \text{rank}(T) = \dim(\text{im}(T)) \). Proposition 2.3.12 and Corollary 2.3.13; image of \( T_A \) is the span of the columns of \( A \). Using echelon form to find bases for \( \text{im}(T_A) \) and \( \ker(T_A) \); see the handout “Finding Bases for Kernel and Image” for more details. Theorem 2.3.17 (Dimension Theorem, or Rank-Nullity Theorem).

- **2.4:** Injective (one-to-one) and surjective (onto). Results relating these concepts to image, kernel, rank, and nullity (pages 96–97, summarized as a five-part corollary in class). Inverse image \( T^{-1}(X) \), and in particular \( T^{-1}(\{w\}) \). Corollary 2.4.17, relating rank and nullity to number of free and basic variables. Proposition 2.4.19, about particular and general solutions.

- **2.5:** Composition of linear maps. All the Propositions and Corollaries in this section. Matrix multiplication, and Proposition 2.5.13. \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \) for a product of matrices \( A, B \). Similarly, \( \text{rank}(TS) \leq \min\{\text{rank}(T), \text{rank}(S)\} \) for a composition of linear maps \( T, S \). See the handout “Some Useful Facts about Linear Transformations” for more details.

- **2.6:** Inverse of a linear map (if the map is bijective). “Isomorphism” and “isomorphic.” Inverse of a matrix and how to compute it (using Gauss-Jordan). Propositions 2.6.7 (that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension) and 2.6.11 (relating the matrix for \( T^{-1} \) to the inverse of the matrix for \( T \)). Exercises 5 and 7 in this section, that \((AB)^{-1} = B^{-1}A^{-1}\) if \( A \) and \( B \) are invertible, and that \((TS)^{-1} = S^{-1}T^{-1}\) if \( S \) and \( T \) are isomorphisms. If \( A \) and \( B \) are square, then either of \( AB = I \) or \( BA = I \) implies the other, so that \( B = A^{-1} \).

- **2.7:** Change of basis matrices. Proposition 2.7.3 and Theorem 2.7.5. Similar matrices. The ideas and methods in the various examples in this section.
Some things you don’t need to know

- Dot product (a.k.a. inner product) and related topics (page 67).
- The rotation map $R_\theta$ and its associated matrix (Examples 2.1.12, 2.2.8, etc.)
- The projection map $P_{\vec{a}}$ and its associated matrix (Examples 2.1.13, 2.2.9, etc.)
- Proposition 2.2.20, that $L(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m\times n}(\mathbb{R})$, whatever that means.
- “Procedure 2” (2.3.15) in Section 2.3.
- Corollaries 2.4.5 and 2.4.9, which use the confusing phrase “only if.” (Very different from “if.”)
- Elementary matrices. (On the other hand, even though I won’t ask you about them, they underlie a lot of what we have done recently, so it’s good to be aware of them.)

Tips

- As before, know all the relevant definitions precisely.
- Know the ideas and themes from the “Thoughts on Proofs” handout, and have the discipline to ignore the assumptions while you structure your proof based solely on what you are trying to prove. (This is why you need to know definitions precisely!) Then try to get from the first line to the last; now you can use the assumptions, but only as tools whenever they are needed.
- All the dimension-counting intuitions and theorems from Chapter 1 have a tendency to come in handy. Still, your main focus should be on Chapter 2.
- If there’s a super-quick way to do a problem (like noticing that the given map $T : V \to W$ cannot possibly be one-to-one, because $\dim W < \dim V$), then do that. On the other hand, don’t overstep the bounds of the theorem you were quoting; if it was $\dim W = \dim V$, then maybe $T$ is one-to-one, or maybe it’s not.
- The rank of a matrix $A \in M_{m\times n}(\mathbb{R})$ can be understood in several different ways and has a lot of cool properties. Make sure you know all of them well:
  - $\text{rank}(A) = \dim(\text{Im}(T_A))$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\vec{x}) = A\vec{x}$.
  - $\text{rank}(A)$ is the dimension of the span of the columns of $A$.
  - $\text{rank}(A)$ is the dimension of the span of the rows of $A$.
  - $\text{rank}(A)$ is the number of pivots of the echelon form of $A$.
  - $\text{rank}(A^t) = \text{rank}(A)$.
  - $T_A$ is surjective if and only if $\text{rank}(A) = m$ (dimension of the target).
  - $T_A$ is injective if and only if $\text{rank}(A) = n$ (dimension of the domain).
  - If $B \in M_{n\times p}(\mathbb{R})$, then $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.
  - If $m = n$, then $T_A$ is invertible (and $A$ is invertible) if and only if $\text{rank}(A) = n$. (Otherwise $\text{rank}(A) < n$.)
  - If $m = n$, then $\text{rank}(A) = n$ if and only if $\det(A) \neq 0$.
- Don’t confuse $T^{-1}(\{\vec{w}\})$ with $T^{-1}(\{\vec{w}\})$. You don’t need $T$ to be invertible to talk about $T^{-1}(\{\vec{w}\})$, for example.
- Unlike the last exam, I don’t plan to ask you to state any definitions or prove any theorems from class. But I will expect you to do some short proofs, so you’ll need to know the definitions and theorems.
- Be careful on the row-reduction and other computational problems. Yes, they’re probably easier (i.e., they require less deep thinking) than proof and theoretical problems, but they are also easy to make silly arithmetic and algebra mistakes on. When possible, know how to check your answer, at least partially (if it’s fairly quick).