Some Tips and Facts about Inner Product Spaces

This handout summarizes some of the main punchlines about inner product spaces, from Sections 4.4, 4.5, and 4.6. I’m not going to review definitions here, and I’ll even skip quite a few of the theorems, even though you need to know those things, too. Instead, my goal in this handout is to summarize the overall storyline, to help you put it all together in your mind.

- When you read, “Let \( V \) be an inner product space,” remember that the primary example is \( V = \mathbb{R}^n \) with the standard inner product (i.e., dot product). There are lots of other examples, but that is the main one to keep in mind.

- That said, when you are trying to prove things about inner product spaces, it is often easier to use the abstract properties of \( \langle \vec{x}, \vec{y} \rangle \) than it is to write out \( \vec{x} = (x_1, \ldots, x_n) \). For example, if \( \vec{x} \) and \( \vec{y} \) are eigenvectors of \( T : V \to V \) with eigenvalues \( \lambda \) and \( \mu \), respectively, and if you’re dealing with \( \langle T\vec{x}, T\vec{y} \rangle \), then instead of writing out coordinates, it’s probably more useful to do something like this:
  \[
  \langle T\vec{x}, T\vec{y} \rangle = \langle \lambda\vec{x}, \mu\vec{y} \rangle = \lambda \langle \vec{x}, \vec{y} \rangle = \lambda \mu \langle \vec{x}, \vec{y} \rangle.
  \]

- Remember that inner products \( \langle \vec{x}, \vec{y} \rangle \) are linear in each coordinate separately. Make sure you know what that means, and be able to use it, as I did in the equation above.

- ONB’s (orthonormal bases) give two key advantages over bases that aren’t orthonormal:
  1. Suppose \( \alpha = \{\vec{v}_1, \ldots, \vec{v}_n\} \) is an ONB for \( V \). For any \( \vec{x} \in V \), if we want to find the coefficients \( c_1, \ldots, c_n \) so that \( \vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \), we can find them fast using the inner product: \( c_i = \langle \vec{x}, \vec{v}_i \rangle \).
    [By contrast, for a garden-variety basis, we’d need to do a Gaussian elimination to find the coefficients.]
  2. Suppose \( \beta = \{\vec{w}_1, \ldots, \vec{w}_k\} \) is an ONB for some subspace \( W \) of \( V \). Then we can write down an explicit formula for the orthogonal projection map \( P_W : V \to V \) onto \( W \):
    \[
    P_W(\vec{v}) = \langle \vec{v}, \vec{w}_1 \rangle \vec{w}_1 + \cdots + \langle \vec{v}, \vec{w}_k \rangle \vec{w}_k
    \]
    So in particular, to find \( [P_W]_{\text{std}} \) (the matrix with respect to the standard basis), compute \( P_W(\vec{e}_1), \ldots, P_W(\vec{e}_n) \) using the above formula, and put the outputs down the columns of a matrix. [See also the next page for another way to compute \( [P_W]_{\text{std}} \).]

- For \( T : V \to V \) linear, the adjoint of \( T \) is just another linear map \( T^* : V \to V \) so that
  \[
  \langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T^*\vec{w} \rangle \quad \text{for all } \vec{v}, \vec{w} \in V.
  \]
  (And yes, such a map \( T^* \) always exists, at least in the case of finite-dimensional \( V \). It’s also always unique.)

- If we are working with \( T : \mathbb{R}^n \to \mathbb{R}^n \) with the standard inner product, then
  \[
  \text{adjoint} = \text{transpose}
  \]
  when using the standard basis for \( \mathbb{R}^n \). More precisely,
  \[
  \text{if we let } A = [T]_{\text{std}} \quad \text{then } A^t = [T^*]_{\text{std}}
  \]
  That said, even when dealing with matrices, it’s usually easier to write equations like \( \langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^t\vec{y} \rangle \) than to write out all the individual entries of \( A \).
• “$T$ is self-adjoint” just means $T = T^*$. So in particular:
  \[ \text{self-adjoint} = \text{symmetric} \]
• Self-adjoint linear maps (again, think symmetric) have nice properties. If $T : V \to V$ is self-adjoint, then:
  • All eigenvalues are real.
  • Eigenvectors corresponding to different eigenvalues are orthogonal.
  • If $\dim(V) < \infty$, then $V$ has an ONB of eigenvectors of $T$.
  • $T$ is a certain linear combination of orthogonal projections (onto eigenspaces, multiplied by the appropriate eigenvalues). See the spectral decomposition (Theorem 4.6.3) for a precise statement.
• Let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an ONB for $\mathbb{R}^n$. Then the change-of-basis matrix $Q = [I]_{\alpha}^{\text{std}}$ has orthonormal columns (namely, the $\vec{v}_i$’s). This gives the cool property that $Q^{-1} = Q^t$. Real matrices with this property are called orthogonal.
• To summarize the last two bullet points:
  • If $A \in M_{n \times n}(\mathbb{R})$ is symmetric, then there is a real diagonal matrix $D$ (of the eigenvalues) and an orthogonal matrix $Q$ (whose columns are the orthonormal eigenvectors) so that $A = QDQ^t$.

(A less important side note): Let $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an ONB for $\mathbb{R}^n$, and let $W$ be the subspace $W = \text{Span}\{\vec{v}_k, \ldots, \vec{v}_\ell\}$. Let $m = \ell - k + 1 = \dim(W)$.

(This situation arises, for example, when $\alpha$ is a basis of eigenvectors of a symmetric matrix $A$, and $\vec{v}_k, \ldots, \vec{v}_\ell$ are the eigenvectors in the list associated with a particular eigenvalue $\lambda$.)

Then the matrix $B = [P_W]_{\alpha}^{\alpha}$ is diagonal, with 1’s in the $k$-th through $\ell$-th entries of the diagonals, and zeros in all other entries. In particular, $B$ is symmetric. In addition, its trace and rank are both $m = \dim(W)$, which is the number of 1’s on the diagonal.

Let $Q$ be the change-of-basis matrix $Q = [I]_{\alpha}^{\text{std}}$, so that the matrix $A$ for $P_W$ with respect to the standard basis is

$$A = [P_W]_{\text{std}}^{\alpha} = [I]_{\alpha}^{\text{std}}[P_W]_{\alpha}^{\alpha}[I]_{\alpha}^{\text{std}} = QBQ^{-1} = QBQ^t,$$

where the last equality is because $Q^{-1} = Q^t$, since $Q$ is an orthogonal matrix. Thus, not only does $A$ also have rank and trace $m$ (since it is similar to $B$), but also, $A$ is symmetric, since $(QBQ^t)^t = QBQ^t$. To summarize, then:

**Fact:** Let $W$ be a subspace of $\mathbb{R}^n$ with $\dim(W) = m$, and let $P_W : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection map onto $W$. Let $A = [P_W]_{\text{std}}$. Then:

• $A$ is symmetric.

• $A$ has rank $m$.

• $A$ has trace $m$. 