Four non-book problems:

1. For the symmetric matrix
   \[
   A = \begin{bmatrix}
   -1 & 0 & 3 \\
   0 & -1 & -1 \\
   3 & -1 & 2
   \end{bmatrix}
   \]
   do the following.
   (a) Find the eigenvalues and eigenvectors.
   (b) Verify that the eigenvalues are all real, and that eigenvectors for different eigenvalues are orthogonal.
   (c) Find an orthonormal basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \).

2. Find the general solution to each of the following systems of differential equations:
   \[
   \begin{align*}
   x'_1 &= 8x_1 + 10x_2 \\
   x'_2 &= -5x_1 - 7x_2
   \end{align*}
   \]
   (a) \( x'_1 = x_1 + x_2 + 2x_3 \)
   \( x'_2 = 2x_1 - 4x_3 \)
   \( x'_3 = x_1 - x_2 \)

3. Find a function \( f(t) \) such that \( f''(t) + 2f'(t) - 8f(t) = 0 \), with \( f(0) = 4 \) and \( f'(0) = 2 \), by the following strategy:
   (a) Define \( x_1(t) = f'(t) \) and \( x_2(t) = f(t) \), and rewrite the original differential equation as
   \[
   \begin{align*}
   x'_1 &= ax_1 + bx_2 \\
   x'_2 &= x_1
   \end{align*}
   \]
   (You will need to figure out what \( a \) and \( b \) are.)
   (b) Find the general solution to the system you wrote down in part (a)
   (c) Use your answer to (b) to find the general solution to the original differential equation.
   (d) Use the initial conditions \( f(0) = 4 \) and \( f'(0) = 2 \) to get the desired particular solution.

4. Fix \( a \in \mathbb{R} \setminus \{0\} \), and let \( A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \). Use induction on \( k \) to prove that for every integer \( k \geq 0 \),
   \[
   A^k = \begin{bmatrix} a^k & ka^{k-1} & \frac{k(k-1)}{2}a^{k-2} \\ 0 & a^k & ka^{k-1} \\ 0 & 0 & a^k \end{bmatrix}.
   \]

(Optional) Challenge problems: 4.6, #1(e), 4(c,d), 10